BOUNDED DERIVATIONS AND THE HOCHSCHILD COHOMOLOGY OF UNIFORM ROE ALGEBRAS

A DISSERTATION SUBMITTED TO THE GRADUATE DIVISION OF THE UNIVERSITY OF HAWAI‘I AT MĀNOA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

May 2021

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Acknowledgments

First, I would like to thank my advisor Professor Rufus Willett. Your patience, guidance, and expertise has been indispensable to my professional and academic development. You encouraged my academic curiosity and let me find my place in the word of mathematics while keeping me on track. I can not thank you enough.

My research was partially funded by the National Science Foundation grants DSM-1564281 and DSM-1901522, for which I am grateful.

I would also like to thank several academics for assisting my exploration of Hochschild cohomology: Professor Stuart White who first suggested that I look into Hochschild cohomology while visiting the University of Hawai‘i, Professor Roger Smith who assisted in the use of his theory through email, and Professor Ilijas Farah for several useful conversations at the Young Mathematicians in C*-Algebras (YMC*A) conference in Copenhagen 2019.

I would also like to thank my committee members and the Graduate Chair, Ruth Haas, for taking time from their busy schedules to assist me.

I would like to thank all of the faculty at the University of Hawai‘i Mathematics Department. Especially Professors Erik Guentner and Karl Heinz Dovermann who shared their wisdom with me through several conversations in the hallways.

Additionally, I would like to thank the Mathematics Department staff. Notably: Sue Hasegawa, Alicia Maedo, and Kim Nishimura who made navigating the bureaucracy and daily life in graduate school possible, and many times enjoyable.

Lastly, I would like to thank my parents, Peggy Lorentz and David Lorentz, Hi MOM!

ABSTRACT

We first show that for a uniform Roe algebra associated to a bounded geometry metric space X, all bounded derivations from that uniform Roe algebra to itself are inner. We obtain this result using a "reduction of cocycles" method from Sinclair and Smith. Then the key technical ingredient comes from recent work of Braga and Farah in their paper "On the Rigidity of Uniform Roe Algebras".

That all bounded derivations are inner is equivalent to the first norm continuous Hochschild cohomology group $H^1_c(C^*_u(X), C^*_u(X))$ vanishing. It is then natural to ask if all the higher groups $H^n_c(C^*_u(X), C^*_u(X))$ vanish. While we cannot answer this question completely, we are able to give necessary and sufficient conditions for the vanishing of $H^n_c(C^*_u(X), C^*_u(X))$.

Lastly, we show that if the norm continuous Hochschild cohomology of a uniform Roe algebra vanishes in all dimensions then the ultraweak-weak* continuous Hochschild cohomology of that uniform Roe algebra vanishes also.

Contents

1 Introduction

The primary objective of this document is to study derivations on, and the Hochschild cohomology of, uniform Roe algebras.

Uniform Roe algebras are a well-studied class of non-separable C^*-algebras associated to metric spaces; see below for basic definitions. They were originally introduced for index-theoretic purposes, but are now studied for their own sake as a bridge between C^*-algebra theory and coarse geometry, as well as having interesting applications to single operator theory and mathematical physics. Due to the presence of $\ell^\infty(X)$ as a diagonal maximal abelian subalgebra, they have a somewhat von Neumann algebraic feel, but are von Neumann algebras only in the trivial finite-dimensional case. Moreover, in many ways they are quite tractable as C^*-algebras, often having good regularity properties such as nuclearity.

In Section 2 we will define uniform Roe algebras and discuss some of their properties. In Section 2 we will also define several different topologies which we shall employ in our study. Since uniform Roe algebras over a space X are a C*-subalgebra of the bounded operators on the square summable sequences over X we may consider several inherited topologies from them such as the weak operator topology and the ultraweak topology.

In Section 3 we will define and study derivations. Motivated by the needs of mathematical physics and the study of one-parameter automorphism groups, it is interesting to study whether all derivations are inner (defined below 3.1.4) for a particular C^*-algebra. In the 1970s, a complete solution to this problem was obtained in the separable case via the work of several authors. The definitive result was obtained by Akemann and Pedersen [2] (see also Elliott [7], which contains a closely related result).

Akemann and Pedersen showed that a separable C^*-algebra only has inner derivations if and only if it isomorphic to a C^*-algebra of the form

$$C \oplus \bigoplus_{i \in I} S_i, \tag{1}$$

where C is continuous trace (possibly zero), and each S_i is simple (possibly zero). In particular, all separable commutative, and all separable simple, C^*-algebras only have inner derivations. However, one might reasonably say that most separable C^*-algebras admit non-inner derivations.

For non-separable C^*-algebras the picture is not as clear. It is well-known that there are non-separable C^*-algebras that are not of the form in line (1) and that only have inner derivations: Most notably, Sakai [18] has shown this for all von Neumann algebras.

Our first goal in this document is to show that uniform Roe algebras only have inner derivations. With this result we have a new class of examples of non-separable C*-algebras that only have inner derivations. Uniform Roe algebras are von Neumann algebras only in the trivial finite-dimensional case. They are also essentially never of the form in line (1).

The first Hochschild cohomology measures how close derivations are to being inner. Hence, our result from Section 3 can be restated as the first Hochschild cohomology of the uniform Roe algebra vanishing. It is then natural to ask if the higher dimensional cohomologies also vanish.

Hochschild cohomology was introduced by Gerhard Hochschild in his 1945 paper *On the Cohomology Groups of an Associative Algebra* [8]. The Hochschild cohomology of associative algebras has become a useful object of study in many fields of mathematics such as representation theory, mathematical physics, and noncommutative geometry, to name a few.

Section 4 will begin with the definition and several properties of multilinear maps which are essential to building the Hochschild complex. We then define the Hochschild complex and Hochschild cohomology as they apply to multilinear maps from a C*-algebra $\mathcal{A}$ to a Banach $\mathcal{A}$-bimodule $\mathcal{V}$. We then review many properties of these cohomologies from Sinclair and Smith's book, *Hochschild cohomology of von Neumann algebras* [19]. The question of whether or not the Hochschild cohomology vanishes in all dimensions for the case of a hyperfinite von Neumann algebra has been answered completely by Kadison and Ringrose.

Definition 1.0.1 (hyperfinite)**.** A von Neumann algebra $\mathcal{M} \subseteq \mathscr{B}(\mathcal{H})$ is *hyperfinite* if there is an increasing family of finite dimensional $*$-subalgebras $\mathcal{M}_\lambda$ whose union is ultraweakly dense in $\mathcal{M}$.

Theorem 1.0.2 ([10] Theorem 3.1)**.** *The Hochschild cohomology of a hyperfinite von Neumann algebra vanishes in all dimensions.*

While we are not able to answer the question of whether or not the Hochschild cohomology vanishes in all dimensions completely for uniform Roe algebras, in Section 5 we are able to give necessary and sufficient conditions for the vanishing of the higher dimensional Hochschild cohomology of a uniform Roe algebra. Specifically, if every element of $H_c^n(C_u^*(X))$ admits a weakly continuous representation, then $H_c^n(C_u^*(X)) = 0$. Note that the converse is trivial.

Lastly, in Section 6, we review the connection between the Hochschild cohomology of ultraweak-weak* continuous multilinear maps and the Hochschild cohomology of norm continuous multilinear maps. We then conclude by showing that if the norm continuous Hochschild cohomology of uniform Roe algebras vanishes in all dimensions then so does the ultraweak-weak* continuous Hochschild cohomology.

2 Preliminaries

Inner products are linear in the first variable. For a Hilbert space $\mathcal{H}$ we denote the space of bounded operators on $\mathcal{H}$ by $\mathscr{B}(\mathcal{H})$, and the space of compact operators by $\mathfrak{K}(\mathcal{H})$.

The Hilbert space of square-summable sequences on a set X is denoted $\ell^2(X)$, and the canonical basis of $\ell^2(X)$ will be denoted $(\vartheta_x)_{x\in X}$ (we reserve δ for derivations). For $a \in \mathscr{B}(\ell^2(X))$ we define its matrix entries by

$$a_{xy} := \langle \vartheta_x, a\vartheta_y \rangle .$$

2.1 Uniform Roe Algebras

We now give some basic definitions regarding uniform Roe algebras.

Definition 2.1.1 (propagation, uniform Roe algebra)**.** Let X be a metric space and $r \geq 0$. An operator $a \in \mathscr{B}(\ell^2(X))$ *has propagation at most* r if $a_{xy} = 0$ whenever $d(x, y) > r$ for all $(x, y) \in X \times X$. In this case, we write $\mathrm{prop}(a) \leq r$. The set of all operators with propagation at most r is denoted $\mathbb{C}_u^r[X]$. We define

$$\mathbb{C}_u[X] := \{a \in \mathscr{B}(\ell^2(X)) : \mathrm{prop}(a) < \infty\};$$

it is not difficult to see that this is a $*$-algebra. The *uniform Roe algebra*, denoted $C_u^*(X)$, is defined to be the norm closure of $\mathbb{C}_u[X]$ under the norm inherited from $\mathscr{B}(\ell^2(X))$.

Definition 2.1.2 (ϵ-r-*approximated*)**.** Let X be a metric space. Given $\epsilon > 0$ and $r > 0$, an operator $a \in \mathscr{B}(\ell^2(X))$ can be ϵ-r-*approximated* if there exists an $b \in \mathbb{C}_u^r[X]$ such that $\|a - b\| \leq \epsilon$. Note that an operator $a \in \mathscr{B}(\ell^2(X))$ is in the uniform Roe algebra if and only if given $\epsilon > 0$ there exists an r such that a can be ϵ-r-approximated.

We will be exclusively interested in uniform Roe algebras associated to bounded geometry metric spaces as in the next definition.

Definition 2.1.3 (bounded geometry)**.** A metric space X is said to have *bounded geometry* if for every $r \geq 0$ there exists an $N_r \in \mathbb{N}$ such that for all $x \in X$, the ball of radius r about x has at most N_r elements.

2.2 Topologies

Throughout this document we will use several different topologies. Since the uniform Roe algebra is a C*-subalgebra of the bounded operators on a Hilbert space we may consider several inherited topologies from them. Moreover, in section 6 we will be working in the enveloping von Neumann algebra for which we will also assign many of the same topologies. While many of these topologies are familiar to those who work with operator algebras, we record them here for completeness.

Definition 2.2.1. (The operator norm) The operator norm for a bounded operator on a Hilbert space is given by

$$\|a\| = \sup_{\|\xi\|=1} \|a\xi\|_{\mathcal{H}}$$

When we refer to the norm topology we mean the metric topology induced by the operator norm.

There will be topologies other than the norm topology that we will be concerned with. For the following definitions recall that for a Banach space X the set of all bounded linear functionals from X to $\mathbb{C}$ is the dual of X denoted X^*.

Definition 2.2.2. (The supremum norm) For a bounded function on a set S to the complex numbers, $f : S \to \mathbb{C}$, we define the supremum norm as

$$\|f\|_\infty = \sup_{x\in S} |f(x)|$$

Definition 2.2.3. (The Y weak topology) Let X be a vector space and let Y be a family of bounded linear functionals on X which separates the points of X. Then the *Y-weak topology on X*, written $\sigma(X, Y)$, is the weakest topology on X for which all the functionals in Y are continuous.

The next topology is a special case of the previous.

Definition 2.2.4 (The weak* topology, $\sigma(X^*, X)$)**.** Let X be a Banach space. Note that X embeds into X^{**} via the natural map $\iota : x \mapsto \hat{x}$ where $\hat{x}(\phi) = \phi(x)$, $\phi \in X^*$. Moreover, the set $\hat{X} = \{\hat{x} : x \in X\}$ separates the points of X^*. The *weak* topology* on X^* is the weakest topology that makes $\hat{x}$ continuous for all $\hat{x} \in \hat{X}$. This means given any open set $U \subseteq \mathbb{C}$, $\hat{x}^{-1}(U)$ is open and such sets generate the topology. Hence, $V \subseteq X^*$ is open if and only if for each $\phi_0 \in V$ there is an $\epsilon > 0$ and there is some finite collection $\{\hat{x}_i\}_{i=1}^n \subseteq \hat{X}$ such that

$$\bigcap_{i=1}^n \{\phi \in X^* : |\hat{x}_i(\phi) - \hat{x}_i(\phi_0)| < \epsilon\} \subseteq V;$$

that is,

$$\bigcap_{i=1}^n \{\phi \in X^* : |\phi(x_i) - \phi_0(x_i)| < \epsilon\} \subseteq V.$$

Moreover, a net $\{\phi_\lambda\}_{\lambda\in\Lambda} \subseteq X^*$ converges weak* to $\phi \in X^*$ if and only if $\phi_\lambda(x) \to \phi(x)$ in $\mathbb{C}$ for each $x \in X$.

Definition 2.2.5 (The weak operator topology, WOT)**.** Let $\mathcal{H}$ be a Hilbert space and let $\mathscr{B}(\mathcal{H})$ be the bounded operators on $\mathcal{H}$. The *weak operator topology* is the topology generated by the family of seminorms $\rho_{\xi,\eta}(a) = |\langle a\xi, \eta\rangle|$, where $\xi, \eta \in \mathcal{H}$ and $a \in \mathscr{B}(\mathcal{H})$. Equivalently, the WOT is the weakest topology that makes that makes linear functionals of the form $\phi_{\xi,\eta}(a) = \langle a\xi, \eta\rangle$ continuous. Thus, a set $V \subseteq \mathscr{B}(\mathcal{H})$ is open if and only if for each $a \in V$ there exists an $\epsilon > 0$ and finite collections $\{\xi_i\}_{i=1}^n, \{\eta_i\}_{i=1}^n \subseteq \mathcal{H}$ such that

$$\bigcap_{i=1}^{n} \{b \in \mathscr{B}(\mathcal{H}) : |\langle (a-b)\xi_i, \eta_i\rangle| < \epsilon\} \subseteq V$$

Moreover, a net $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq \mathscr{B}(\mathcal{H})$ converges weakly to $a \in \mathscr{B}(\mathcal{H})$ if and only if $\langle a_\lambda\xi, \eta\rangle \to \langle a\xi, \eta\rangle$ in $\mathbb{C}$ for each $\xi, \eta \in \mathcal{H}$.

Definition 2.2.6 (Strong Operator Topology, SOT)**.** Let $\mathcal{H}$ be a Hilbert space and let $\mathscr{B}(\mathcal{H})$ be the bounded operators on $\mathcal{H}$. The *strong operator topology* is the topology generated by the family of seminorms $\rho_\xi(a) = \|a\xi\|$, where $\xi \in \mathcal{H}$ and $a \in \mathscr{B}(\mathcal{H})$. Equivalently, the SOT is the weakest topology such that the evaluation maps $\mathrm{ev}_\xi(a) = a\xi$ are continuous. Thus, a set $V \subseteq \mathscr{B}(\mathcal{H})$ is open if and only if for each $a \in V$ there exists an $\epsilon > 0$ and a finite collection $\{\xi_i\}_{i=1}^n \subseteq \mathcal{H}$ such that

$$\bigcap_{i=1}^{n} \{b \in \mathscr{B}(\mathcal{H}) : \|(a-b)\xi_i\| < \epsilon\} \subseteq V$$

Moreover, a net $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq \mathscr{B}(\mathcal{H})$ converges strongly to $a \in \mathscr{B}(\mathcal{H})$ if and only if $a_\lambda\xi \to a\xi$ in $\mathcal{H}$ for each $\xi \in \mathcal{H}$.

The next topology we shall use is called the ultraweak topology. One must carefully note that in general the ultraweak topology is stronger than the weak operator topology. We will be using this topology exclusively on von Neumann algebras and since every von Neumann algebra has a unique predual by a theorem of Sakai [16], we are able to give the following definition.

Definition 2.2.7 (Ultraweak topology)**.** The *ultraweak topology* of a von Neumann algebra $\mathcal{M}$ is the weak* topology induced by the isometric isomorphism $(\mathcal{M}_*)^* \cong \mathcal{M}$ where $\mathcal{M}_*$ is the unique predual of $\mathcal{M}$.

2.3 Functional Analysis Details

Lastly, in this subsection we prove some facts from functional analysis that we will need.

Proposition 2.3.1. Let $T, S \in \mathscr{B}(\mathcal{H})$ be such that $\|T - S\| > \epsilon$. Then there exist unit vectors $\xi, \eta \in \mathcal{H}$ such that $|\langle (T-S)\xi, \eta\rangle| > \epsilon$.

Proof. Since $T - S \in \mathscr{B}(\mathcal{H})$,

$$\|T - S\| = \sup_{\|\zeta\|=1} \|(T-S)\zeta\| = M > \epsilon, \text{ for some } M \in \mathbb{R}.$$

Thus, there exists a $\xi \in \mathcal{H}$, $\|\xi\| = 1$ such that

$$\|(T-S)\xi\| + (M - \epsilon) > \sup_{\|\zeta\|=1} \|(T-S)\zeta\| = M,$$

so that $\|(T-S)\xi\| > \epsilon$. Next, $\|(T-S)\xi\| = L > \epsilon$ for some $L \in \mathbb{R}$. Thus, $|\langle (T-S)\xi, (T-S)\xi\rangle| = L^2$ so that

$$\left|\left\langle (T-S)\xi, \frac{(T-S)\xi}{\|(T-S)\xi\|}\right\rangle\right| = \frac{L^2}{\|(T-S)\xi\|} = L > \epsilon.$$

Taking $\eta = \frac{(T-S)\xi}{\|(T-S)\xi\|}$ we have that $|\langle (T-S)\xi, \eta\rangle| > \epsilon$ for the unit vectors $\xi, \eta \in \mathcal{H}$. □

Proposition 2.3.2. For some index set I, let $\{a_i\}_{i\in I} \subseteq \mathscr{B}(\mathcal{H})$ be a uniformly bounded net weakly converging to an operator $a \in \mathscr{B}(\mathcal{H})$. Then for any compact operators $k, q \in \mathfrak{K}(\mathcal{H})$ the net $\{ka_iq\}_{i\in I}$ converges in norm to kaq.

Proof. Suppose not. Then there exists an $\epsilon > 0$ such that for all $i_0 \in I$ there exists $i \geq i_0$ such that $\|k(a_i)q\| \geq \epsilon$ where $a_i \overset{\text{wot}}{\to} 0$. Thus, by Proposition 2.3.1, for each such a_i there exist unit vectors $\xi_i,\ \eta_i$ such that

$$|\langle k a_i q \xi_i, \eta_i\rangle| = |\langle a_i q \xi_i, k\eta_i\rangle| \geq \frac{\epsilon}{2}.$$

Next, since $k, q \in \mathfrak{K}(\mathcal{H})$ and all of the ξ_i's and η_i's are unit vectors, there exists convergent subnets

$$q\xi_\alpha \to \xi,\ \|q\| \geq \|\xi\| \quad \text{and} \quad k\eta_\beta \to \eta,\ \|k\| \geq \|\eta\|$$

Thus, letting M be a uniform bound on the a_i's, there exists an α_0 such that

$$\|q\xi_\alpha - \xi\| \leq \min\left\{\frac{\epsilon}{12M\,\|k\|}, \frac{\epsilon}{12M\,\|q\|}, \frac{\epsilon}{12}, 1\right\} \quad \text{whenever} \quad \alpha \geq \alpha_0,$$

and in turn a $\beta_0 \geq \alpha_0$ such that

$$\|k\eta_\beta - \eta\| \leq \min\left\{\frac{\epsilon}{12M\,\|k\|}, \frac{\epsilon}{12M\,\|q\|}, \frac{\epsilon}{12}, 1\right\}, \quad \text{whenever} \quad \beta \geq \beta_0.$$

Additionally, since $a_i \overset{\text{wot}}{\to} 0$ there exists a $i_{\xi,\eta} \geq \beta_0$ such that

$$|\langle a_i \xi, \eta\rangle| < \frac{\epsilon}{4} \quad \text{whenever} \quad i \geq i_{\xi,\eta}.$$

Then, putting this together, we have that

$$|\langle a_i q\xi_i, k\eta_i\rangle| = |\langle a_i(q\xi_i - \xi + \xi), (k\eta_i - \eta + \eta)\rangle|$$

$$\leq |\langle a_i\xi, \eta\rangle| + |\langle a_i\xi, (k\eta_i - \eta)\rangle| + |\langle a_i(q\xi_i - \xi), \eta\rangle| + |\langle a_i(q\xi_i - \xi), (k\eta_i - \eta)\rangle|$$
$$< \frac{\epsilon}{2}$$

a contradiction. $\square$

Proposition 2.3.3. Let $(T_\alpha)_{\alpha\in A}$ be a uniformly bounded net in $\mathscr{B}(\mathcal{H})$ converging strongly to $T \in \mathscr{B}(\mathcal{H})$. Then for any compact operator $k \in \mathfrak{K}(\mathcal{H})$, the net $(T_\alpha k)_{\alpha\in A}$ converges in norm.

Proof. Suppose for contradiction that $(T_\alpha k)_{\alpha\in A}$ does not converge in norm. Let M be a uniform bound on $(T_\alpha k)_{\alpha\in A}$. Then there exists an $\epsilon > 0$ such that for all $j \in A$ there exists an $\alpha_j \geq j$ such that $\left\|T_{\alpha_j} k - Tk\right\| \geq \epsilon$. Hence, for all $j \in A$ there exists a $\xi_{\alpha_j} \in \ell^2(X)$, $\left\|\xi_{\alpha_j}\right\| = 1$ such that $\left\|(T_{\alpha_j} k - Tk)\xi_{\alpha_j}\right\| > \frac{\epsilon}{2}$.

Next, since k is finite rank, the image of the closed unit ball of $\ell^2(X)$ is compact under k. Thus, the net $(k\xi_{\alpha_j})_{j\in A}$ has a convergent subnet, $k\xi_\beta \to \eta$. Then, since $(T_\alpha)_{\alpha\in A}$ converges strongly there exists an α_η such that

$$\|(T_\alpha - T)\eta\| < \tfrac{\epsilon}{4} \text{ whenever } \alpha \geq \alpha_\eta.$$

Moreover, there exists a β_0 such that

$$\|k\xi_\beta - \eta\| < \tfrac{\epsilon}{4(M+\|T\|)} \text{ whenever } \beta \geq \beta_0.$$

Taking $\beta \geq \alpha_\eta,\ \beta \geq \beta_0$ we have

$$\|(T_\beta k - Tk)\xi_\beta\| = \|(T_\beta - T)(k\xi_\beta - \eta + \eta)\|$$

$$= \|(T_\beta - T)(\eta) + (T_\beta - T)(k\xi_\beta - \eta)\|$$

$$\leq \|(T_\beta - T)(\eta)\| + (\|T_\beta\| + \|T\|)\,\|k\xi_\beta - \eta\| \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$$

contradicting that $\|(T_\beta k - Tk)\xi_\beta\| > \frac{\epsilon}{2}$. □

Proposition 2.3.4. Suppose that: $(P_j)_{j\in J}$ is an increasing net of finite rank projections in $\ell^\infty(X)$ converging strongly to the identity, $(S_j)_{j\in J}$ is a net in $\mathscr{B}(\mathcal{H})$ converging weakly to S, and for the unit vectors $\xi, \eta \in \ell^2(X)$ we have that $|\langle (T-S)\xi, \eta\rangle| > \epsilon$ for some fixed ϵ. Then there exists a j_0 such that $\|(TP_j - S_j)\xi\| > \epsilon$ whenever $j \geq j_0$.

Proof. First note that, since P_j converges strongly to the identity, $P_j\xi \to \xi$ in norm. Thus,

$$\lim_{j\in J} \langle (TP_j - S_j)\xi, \eta\rangle = \lim_{j\in J} \langle TP_j\xi, \eta\rangle - \lim_{j\in J} \langle S_j\xi, \eta\rangle = \langle (T - S)\xi, \eta\rangle$$

for some $M \in \mathbb{R}$. Hence, there exists a j_0 such that for all $j \geq j_0$ we have that

$$|\langle (TP_j - S_j)\xi, \eta\rangle - \langle (T - S)\xi, \eta\rangle| < M - \epsilon$$

so that

$$M = |\langle (T - S)\xi, \eta\rangle| < |\langle (TP_j - S_j)\xi, \eta\rangle| + (M - \epsilon) \implies |\langle (TP_j - S_j)\xi, \eta\rangle| > \epsilon$$

whenever $j \geq j_0$. Lastly, since $\|\eta\| = 1$, and by the Cauchy Schwarz inequality,

$$\|(TP_j - S_j)\xi\|^2 = \langle (TP_j - S_j)\xi, (TP_j - S_j)\xi\rangle \langle \eta, \eta\rangle \geq |\langle (TP_j - S_j)\xi, \eta\rangle|^2 > \epsilon^2.$$

Thus, taking roots on both sides yields the desired result. $\square$

3 Bounded Derivations on Uniform Roe Algebras

Note that much of the material in this section has been adapted from Rufus Willett's and the author's paper *Bounded derivations on uniform Roe algebras* [12]. Here is the main theorem of this section.

Theorem 3.0.1. *Uniform Roe algebras associated to bounded geometry metric spaces only have inner derivations.*

The key ingredients in the proof are: a basic form of a 'reduction of cocycles' argument used by Sinclair and Smith (cf. Remark 3.2.2 [19]) in their study of Hochschild cohomology of von Neumann algebras; and recent applications of Ramsey-theoretic ideas to the study of uniform Roe algebras by Braga and Farah (Lemma 4.9 [5]).

3.1 Derivations

Definition 3.1.1. Let $\mathcal{A}$ be a C^*-algebra. A *derivation* of A is a linear map $\delta : A \to A$ satisfying $\delta(ab) = a\delta(b) + \delta(a)b$.

In this document, we always assume that our derivations are defined on all of $\mathcal{A}$, and are thus bounded by a fundamental result of Sakai [17]. We denote the commutator of $a, b \in \mathscr{B}(\mathcal{H})$ by $[a, b] := ab - ba$.

Definition 3.1.2 (spatial derivation)**.** Let $\mathcal{A} \subseteq \mathscr{B}(\mathcal{H})$ be a concrete C^*-algebra. A derivation δ of $\mathcal{A}$ is *spatial* if there is a bounded operator $d \in \mathscr{B}(\mathcal{H})$ such that $\delta(a) = [a, d]$.

The following is due to Kadison [9, Theorem 4].

Theorem 3.1.3. *Let $\mathcal{A} \subseteq \mathscr{B}(\mathcal{H})$ be a concrete C^*-algebra. Then every derivation on $\mathcal{A}$ is spatial.*

Definition 3.1.4. A derivation δ of $\mathcal{A}$ is *inner* if there exists d in the multiplier algebra $M(\mathcal{A})$ of $\mathcal{A}$ such that $\delta(a) = [a, d]$ for all $a \in \mathcal{A}$. Let us say that a C*-algebra $\mathcal{A}$ *only has inner derivations* if all (bounded) derivations are inner.

Note that the difference between a spatial derivation and an inner derivation is the location of the element that induces the derivation. That is, if $\delta(a) = [a, d]$ is a derivation on a concrete C*-algebra $\mathcal{A} \subseteq \mathscr{B}(\mathcal{H})$ where $a \in \mathcal{A}$ and $d \in \mathscr{B}(\mathcal{H})$, then δ is spatial by definition. If further, $d \in \mathcal{A}$, then δ is inner.

3.2 Averaging over Amenable Groups

In this subsection, we summarize some facts we need about averaging operators over an amenable group. We shall use this averaging process in this section to prove Theorem 3.0.1. In the sequel we will use this averaging process to apply Sinclair and Smith's 'reduction of cocycles' technique.

Let G be a discrete (possibly uncountable) group. If A is a complex Banach space, we let $\ell^\infty(G, A)$ denote the Banach space of bounded functions from G to A equipped with the supremum norm; in the case $A = \mathbb{C}$, we just write $\ell^\infty(G)$. We also equip $\ell^\infty(G, A)$ with the right-action of G defined for $a \in \ell^\infty(G, A)$ and $h, g \in G$ by

$$(ag)(h) := a(hg^{-1}).$$

If Z is any set, a function $\phi : \ell^\infty(G, A) \to Z$ is *invariant* if $\phi(ag) = \phi(a)$ for all $a \in \ell^\infty(G, A)$ and $g \in G$.

Recall that G is amenable if there exists an *invariant mean* on $\ell^\infty(G)$, i.e. an invariant function $\Phi : \ell^\infty(G) \to \mathbb{C}$ that is also a state. Fix an invariant mean Φ on $\ell^\infty(G)$ and let B be a complex Banach space with dual B^*. We may upgrade an invariant mean on $\ell^\infty(G)$ to an invariant contractive linear map $\ell^\infty(G, B^*) \to B^*$ in the following way. Let $b \in B$, $g \in G$, and

$a \in \ell^\infty(G, B^*)$, and write $\langle b, a(g)\rangle$ for the pairing between b and $a(g)$. Define a map

$$\Psi_{b,a} : G \to \mathbb{C} \ \text{ by } \ g \mapsto \langle b, a(g)\rangle \,.$$

Note that $|\Psi_{b,a}(g)| = |\langle b, a(g)\rangle| \leq \|a\|_{\ell^\infty(G,B^*)} \|b\|_B$ for all $g \in G$. Hence, $\Psi_{b,a} \in \ell^\infty(G)$ for all $b \in B$ and for all $a \in \ell^\infty(G, B^*)$ so that when we apply Φ we get a complex number $\Phi(\Psi_{b,a})$. We now define a map

$$\Phi_a : B \to \mathbb{C} \ \text{ by } \ b \mapsto \Phi(\Psi_{b,a}).$$

Observe that, since Φ is a state,

$$|\Phi_a(b)| = |\Phi(\Psi_{b,a})| \leq \|\Psi_{b,a}\|_{\ell^\infty(G)} \leq \|a\|_{\ell^\infty(G,B^*)} \|b\|_B \tag{2}$$

and so $\Phi_a \in B^*$. Lastly, we define

$$\Psi : \ell^\infty(G, B^*) \to B^* \ \text{ by } \ a \mapsto \Phi_a.$$

Lemma 3.2.1. *With notation as above, the map*

$$\Psi : \ell^\infty(G, B^*) \to B^* \ \textit{ defined by } \ a \mapsto \Phi_a$$

is uniquely determined by the condition

$$\langle b, \Psi(a)\rangle = \Phi(\langle b, a(\cdot)\rangle) \tag{3}$$

for $b \in B$ and $a \in \ell^\infty(G, B^*)$. It is contractive, linear, invariant, and acts as the identity on constant functions.

Proof. To show uniqueness let $\Theta : \ell^\infty(G, B^*) \to B^*$ be a map that also satisfies

$$\langle b, \Theta(a)\rangle = \Phi(\langle b, a(\cdot)\rangle) \ \text{ for all } \ b \in B, \ a \in \ell^\infty(G, B^*).$$

Let $a \in \ell^\infty(G, B^*)$ be fixed but arbitrary. Then

$$\langle b, \Theta(a) \rangle = \Phi(\langle b, a(\cdot) \rangle) = \langle b, \Psi(a) \rangle \text{ for all } b \in B.$$

Thus, $\Theta(a) = \Psi(a)$ and since a was arbitrary $\Theta = \Psi$.

Note that

$$\|\Psi(a)\| = \|\Phi_a\| \leq \|a\|_{\ell^\infty(G,B^*)}$$

by (2) and so Ψ is contractive.

To see that Ψ is linear let $b \in B$, $a, a' \in \ell^\infty(G, B^*)$, and $\lambda \in \mathbb{C}$. Note that

$$\langle b, (a + \lambda a')(g) \rangle = \langle b, a(g) \rangle + \lambda \langle b, a'(g) \rangle \text{ for all } g \in G$$

so by (3) we have

$$\langle b, \Psi(a + \lambda a') \rangle = \Phi(\Psi_{b,(a+\lambda a')}) = \Phi(\Psi_{b,a}) + \lambda \Phi(\Psi_{b,a'}) = \langle b, \Psi(a) \rangle + \lambda \langle b, \Psi(a') \rangle$$

$$= \langle b, \Psi(a) + \lambda \Psi(a') \rangle$$

and so $\Psi(a + \lambda a') = \Psi(a) + \lambda \Psi(a')$ since b was arbitrary.

Next recall that Ψ is invariant for G if $\Psi(ag) = \Psi(a)$ for all $g \in G$ and all $a \in \ell^\infty(G, B^*)$. Let $g, h \in G$ and observe that

$$\Psi_{b,ag}(h) = \langle b, ag(h) \rangle = \left\langle b, a(hg^{-1}) \right\rangle = \Psi_{b,a}(hg^{-1}) = (\Psi_{b,a}g)(h)$$

so by the invariance of Φ we have

$$\langle b, \Psi(ag) \rangle = \Phi(\Psi_{b,ag}) = \Phi(\Psi_{b,a}g) = \Phi(\Psi_{b,a}) = \langle b, \Psi(a) \rangle$$

for all $b \in B$. Thus, $\Psi(ag) = \Psi(a)$ so that Ψ is invariant for G.

Lastly, suppose that $a \in \ell^\infty(G, B^*)$ is constant. That is, $a(g) = v_0 \in B^*$ for all $g \in G$. Then, since Φ is a state, we have

$$\langle b, \Psi(a) \rangle = \Phi(\Psi_{b,a}) = \Phi(\langle b, v_0 \rangle\, 1_{\ell^\infty(G)}) = \langle b, v_0 \rangle\, \Phi(1_{\ell^\infty(G)}) = \langle b, v_0 \rangle$$

and so $\Psi(a) = v_0$. $\square$

Before we conclude with the properties of Ψ we will introduce an action by a C*-algebra $\mathcal{A}$ on B^*. We then 'upgrade' this action to an action on $\ell^\infty(G, B^*)$ and B. Once this is done we will be able to show that Ψ behaves 'like' a conditional expectation. That is, for $x, y \in \mathcal{A}$, $f \in \ell^\infty(G, B^*)$, $\Psi(x \cdot f \cdot y) = x \cdot \Psi(f) \cdot y$. First, we will need a few definitions and lemmas.

Definition 3.2.2. (Nondegenerate action) We say that a $*$-algebra $\mathcal{A}$ acts *nondegenerately* on a left (resp. right) $\mathcal{A}$-module $\mathcal{V}$ if $\overline{\mathcal{A}\mathcal{V}} = \mathcal{V}$.

Definition 3.2.3 (Banach $\mathcal{A}$-bimodule)**.** Let $\mathcal{A}$ be a C*-algebra. We say that $\mathcal{V}$ is a *Banach $\mathcal{A}$-bimodule* if $\mathcal{A}$ acts nondegenerately on $\mathcal{V}$ from both the left and the right and $\mathcal{V}$ has a norm under which it is a Banach space. Moreover, the norm on $\mathcal{V}$ satisfies

$$\|av\|_{\mathcal{V}} \le \|a\|_{\mathcal{A}} \|v\|_{\mathcal{V}} \quad \text{and} \quad \|va\|_{\mathcal{V}} \le \|v\|_{\mathcal{V}} \|a\|_{\mathcal{A}} \quad \text{for all} \quad a \in \mathcal{A},\ v \in \mathcal{V}.$$

Lemma 3.2.4. *Let $\mathcal{A}$ be a C^*-algebra and suppose that B^* is a Banach $\mathcal{A}$-bimodule. Then we can make $\ell^\infty(G, B^*)$ a Banach $\mathcal{A}$-bimodule via*

$$(x \cdot f)(g) := x \cdot f(g), \quad \textit{and} \quad (f \cdot x)(g) := f(g) \cdot x$$

where $f \in \ell^\infty(G, B^*)$, $x \in \mathcal{A}$, and $g \in G$.

Proof. Let $f, f' \in \ell^\infty(G, B^*)$, and $x, y \in \mathcal{A}$. First note that,

$$\|x \cdot f\|_{\ell^\infty} = \sup_{g\in G} \|x \cdot f(g)\|_{B^*} \leq \|x\|_{\mathcal{A}} \sup_{g\in G} \|f(g)\|_{B^*} = \|x\|_{\mathcal{A}} \|f\|_{\ell^\infty}$$

and

$$\|f \cdot x\|_{\ell^\infty} = \sup_{g\in G} \|f(g) \cdot x\|_{B^*} \leq \sup_{g\in G} \|f(g)\|_{B^*} \|x\|_{\mathcal{A}} = \|f\|_{\ell^\infty} \|x\|_{\mathcal{A}}.$$

Next, observe that for all $g \in G, \lambda \in \mathbb{C}$

1. $(x \cdot (\lambda f + f'))(g) = x \cdot ((\lambda f + f')(g)) = x \cdot (\lambda f(g) + f'(g)) = x \cdot \lambda f(g) + x \cdot f'(g) = (\lambda x \cdot f + x \cdot f')(g)$
2. $((\lambda x + y) \cdot f)(g) = (\lambda x + y) \cdot (f(g)) = \lambda x \cdot f(g) + y \cdot f(g) = (\lambda x \cdot f)(g) + (y \cdot f)(g) = (\lambda x \cdot f + y \cdot f)(g)$, and
3. $((xy) \cdot f)(g) = (xy) \cdot f(g) = x \cdot (y \cdot f(g)) = x \cdot ((y \cdot f)(g)) = (x \cdot (y \cdot f))(g)$

with similar calculations when $\mathcal{A}$ acts from the right.

Thus, $\ell^\infty(G, B^*)$ is a Banach $\mathcal{A}$-bimodule. □

Before we upgrade the action of $\mathcal{A}$ on B^*, to an action on B we will need the following theorem.

Theorem 3.2.5 ([15] Theorem IV.20)**.** *The $\sigma(X, Y)$ continuous linear functionals on X are precisely Y.*

Proof. Let ℓ be an arbitrary $\sigma(X, Y)$-continuous linear functional on X. Note that the set $L_1 := \{x \in X : |\ell(x)| < 1\}$ is a $\sigma(X, Y)$ open set in X since ℓ is continuous in this topology. Moreover, $0 \in L_1$, and so there exists $\{y_i\}_{i=1}^n \subseteq Y$ and a $\epsilon > 0$ such that

$$L_1 \supseteq \{x \in X : |y_i(x)| < \epsilon, i \in \{1, \ldots, n\}\}.$$

Next, let $K := \{x \in X : y_i(x) = 0, i \in \{1, \dots, n\}\}$ so that $L_1 \supseteq K$. Note that, $\gamma x \in K$ whenever $x \in K$ for all $\gamma \in \mathbb{C}$ by the linearity of the y_i's. Thus, for any $x \in K, \gamma > 0$ we have,

$$\left|\ell(\gamma^{-1}x)\right| < 1 \iff |\ell(x)| < \gamma.$$

Letting $\gamma \to 0$, we see that $\ell(x) = 0$ for all $x \in K$. Thus, we may define $\tilde{\ell}$ on X/K by $\tilde{\ell}(x + K) = \ell(x)$. Note that, by [11] Proposition 1.1.1,

$$\tilde{\ell} = \sum_{i=1}^{n} \alpha_i \tilde{y}_i \text{ for some } \alpha_i \in \mathbb{C}, \text{ since span}\{y_1, \dots, y_n\} = (X/K)^*.$$

Thus, $\ell = \sum_{i=1}^{n} \alpha_i y_i \in Y$. Since ℓ was arbitrary, we are done. □

Lemma 3.2.6. *Let $\mathcal{A}$ be a C^*-algebra and suppose that B^* is a Banach $\mathcal{A}$-bimodule such that for $b^* \in B^*$ and $x \in \mathcal{A}$ the maps*

$$L_x : b^* \mapsto x \cdot b^* \quad \textit{and} \quad R_x : b^* \mapsto b^* \cdot x$$

are weak continuous. Then we can make B an $\mathcal{A}$-bimodule via actions that satisfy*

$$\langle x \cdot b, b^* \rangle = \langle b, x^* \cdot b^* \rangle \quad \textit{and} \quad \langle b \cdot x, b^* \rangle = \langle b, b^* \cdot x^* \rangle \quad \textit{where} \;\; b \in B.$$

Proof. First, we dualize B^* with respect to the $\sigma(B^*, B)$ topology which we denote by $B^{*\dagger}$. Note that the topology on $B^{*\dagger}$ is the weakest topology that makes the evaluation maps $\mathrm{ev}_b : b^* \to \mathbb{C}$ continuous. Moreover, by Theorem 3.2.5, $B^{*\dagger} \cong B$. Thus, dualizing the maps L_x and R_x with respect to the $\sigma(B^*, B)$ topology the maps $L_x^\dagger$ and $R_x^\dagger$ are maps on B for all $x \in \mathcal{A}$.

Next, let $x, y \in \mathcal{A}$, $b^* \in B^*$, $\lambda \in \mathbb{C}$, and let $b, c \in B$. Observe that,

1. $\langle x \cdot (\lambda b + c), b^* \rangle = \langle (\lambda b + c), x^* \cdot b^* \rangle = \langle \lambda b, x^* \cdot b^* \rangle + \langle c, x^* \cdot b^* \rangle = \langle x \cdot \lambda b, b^* \rangle + \langle x \cdot c, b^* \rangle = \langle (\lambda x \cdot b + x \cdot c), b^* \rangle$

2. $\langle (\lambda x + y) \cdot b, b^* \rangle = \langle b, (\lambda x + y)^* \cdot b^* \rangle = \left\langle b, \overline{\lambda} x^* \cdot b^* \right\rangle + \langle b, y^* \cdot b^* \rangle = \langle \lambda x \cdot b, b^* \rangle + \langle y \cdot b, b^* \rangle = \langle (\lambda x \cdot b + y \cdot b), b^* \rangle$, and

3. $\langle (xy) \cdot b, b^* \rangle = \langle b, (xy)^* \cdot b^* \rangle = \langle b, y^* \cdot (x^* \cdot b^*) \rangle = \langle y \cdot b, x^* \cdot b^* \rangle = \langle x \cdot (y \cdot b), b^* \rangle$

with similar calculations when $\mathcal{A}$ acts on the right. Thus, the action defined above is a well defined action on B.

□

Lemma 3.2.7. *Let $\mathcal{A}$ be a C^*-algebra and suppose that B^* is a Banach $\mathcal{A}$-bimodule such that for $b^* \in B^*$ and $x \in \mathcal{A}$ the maps*

$$b^* \mapsto x \cdot b^* \quad and \quad b^* \mapsto b^* \cdot x$$

are weak continuous. Then the averaging operator $\Psi : \ell^\infty(G, B^*) \to B^*$ as defined above has the property that*

$$\Psi(x \cdot f) = x \cdot \Psi(f) \quad and \quad \Psi(f \cdot x) = \Psi(f) \cdot x$$

Proof. Let $b \in B,\ f \in \ell^\infty(G, B^*)$, and $x \in \mathcal{A}$. Observe that

$$\langle b, x \cdot \Psi(f) \rangle = \langle x^* \cdot b, \Psi(f) \rangle = \Phi(\langle x^* \cdot b, f(\cdot) \rangle) = \Phi(\langle b, (x \cdot f)(\cdot) \rangle) = \langle b, \Psi(x \cdot f) \rangle$$

with a similar calculation when $\mathcal{A}$ acts on the right. □

Initially we will be using this averaging process in conjunction with the commutator bracket. However, in the sequel we will also be using this machinery to average over multilinear maps. Rather then defining new maps for each situation, and since our averaging operator enjoys all of the properties (except for countable additivity) as if we were integrating over a normalized Haar measure, we will use integral notation to denote our averaging operator. That is, if Ψ is as above for $a \in \ell^\infty(G, B^*)$ and $g \in G$ we define

$$\Psi(a) =: \int_G a(g) \, \mathrm{d}\, \mu(g).$$

Note that, in the non-compact amenable case, μ is not a measure; it serves only as a notational device. For completeness we enumerate the properties of the averaging operator in integral notation below. Let $a, b \in \ell^\infty(G, B^*)$, $g, g' \in G$, $v \in B^*$, and $\lambda \in \mathbb{C}$ we have:

1. linear

$$\int_G (a + \lambda b)(g) \, \mathrm{d}\, \mu(g) = \int_G a(g) \, \mathrm{d}\, \mu(g) + \lambda \int_G b(u) \, \mathrm{d}\, \mu(g),$$

2. contractive

$$\left\| \int_G a(g) \, \mathrm{d}\, \mu(g) \right\|_{B^*} \leq \|a\|_{\ell^\infty(G,B^*)},$$

3. invariant

$$\int_G ag(g') \, \mathrm{d}\, \mu(g') = \int_G a(g') \, \mathrm{d}\, \mu(g'), \text{ and}$$

4. acts as the identity on constant functions

$$\int_G v \, \mathrm{d}\, \mu(g) = v.$$

5. Lastly, if B^* is an $\mathcal{A}$-bimodule for a C*-algebra $\mathcal{A}$ and the action is weak* continuous as in Lemma 3.2.7, then the averaging operator is a $\mathcal{A}$-bimodular map

$$\int_G xa(g)y \, \mathrm{d}\, \mu(g) = x \left(\int_G a(g) \, \mathrm{d}\, \mu(g) \right) y \quad x, y \in \mathcal{A}.$$

We will apply this machinery in the case that $B = \mathcal{L}^1(\ell^2(X))$ is the trace class operators on $\ell^2(X)$. In this case, the dual B^* canonically identifies with $\mathscr{B}(\ell^2(X))$: indeed, if Tr is the canonical trace on $\mathcal{L}^1(\ell^2(X))$, $b \in \mathcal{L}^1(\ell^2(X))$, and $a \in \mathscr{B}(\ell^2(X))$, then the pairing inducing this duality isomorphism is defined by

$$\langle b, a \rangle := \operatorname{Tr}(ba). \tag{4}$$

The next lemma says that our averaging process behaves well with respect to propagation. The main point of the lemma is that the collection of operators in $\mathscr{B}(\ell^2(X))$ that have propagation at most r is weak-$*$ closed for the weak-$*$ topology inherited from the pairing with $\mathcal{L}^1(\ell^2(X))$.

Lemma 3.2.8. *With notation as above, if $r \geq 0$ and $a \in \ell^\infty(G, \mathscr{B}(\ell^2(X)))$ is such that the propagation of each $a(g)$ is at most r for all $g \in G$, then the propagation of $\int_G a(g) \,\mathrm{d}\,\mu(g)$ is also at most r.*

Proof. Let $e_{xy} \in \mathcal{L}^1(\ell^2(X))$ be the standard matrix unit. Then one computes using line (4) above that for any $a \in \mathscr{B}(\ell^2(X))$,

$$\langle e_{yx}, a \rangle = \operatorname{Tr}(e_{yx} a) = a_{xy}. \tag{5}$$

Using lines (3) and (5), we see that

$$\left\langle e_{yx}, \int_G a(g) \,\mathrm{d}\,\mu(g) \right\rangle = \int_G \langle e_{yx}, a(g) \rangle \,\mathrm{d}\,\mu(g) = \int_G a(g)_{xy} \,\mathrm{d}\,\mu(g),$$

where the last expression means the image of the function

$$G \to \mathbb{C}, \quad g \mapsto a(g)_{xy} \text{ under the invariant mean.}$$

Now, if $d(x, y) > r$, we have that $a(g)_{xy} = 0$ for all $g \in G$, and therefore that $\int_G a(g)_{xy} \, \mathrm{d}\, \mu(g) = 0$. Hence, by the above computation,

$$d(x, y) > r \quad \text{implies} \quad \left\langle e_{yx}, \int_G a(g) \, \mathrm{d}\, \mu(g) \right\rangle = 0.$$

Using line (5), this says that $\int_G a(g) \, \mathrm{d}\, \mu(g)$ has propagation at most r, so we are done. $\square$

Lemma 3.2.9. *With notation as above, say that there is a unitary representation $g \mapsto u_g$ of G on $\ell^2(X)$. For any fixed $d \in \mathscr{B}(\ell^2(X))$, define $a \in \ell^\infty(G, \mathscr{B}(\ell^2(X)))$ by $a(g) := u_g^* d u_g$. Then $\int_G a(g) \, \mathrm{d}\, \mu(g)$ is in the commutant of the set $\{u_g \mid g \in G\}$.*

Proof. Let $h \in G$. Then by Lemma 3.2.7,

$$u_h \int_G u_g^* d u_g \, \mathrm{d}\, \mu(g) = \int_G u_h u_g^* d u_g \, \mathrm{d}\, \mu(g) = \int_G u_{gh^{-1}}^* d u_g \, \mathrm{d}\, \mu(g).$$

Making the 'change of variables' $k = gh^{-1}$ and using right-invariance of the map $a \mapsto \int_G a(g) \, \mathrm{d}\, \mu(g)$, this equals

$$\int_G (u_k)^* d u_{kh} \, \mathrm{d}\, \mu(k) = \int_G u_k^* d u_k u_h \, \mathrm{d}\, \mu(k).$$

Using Lemma 3.2.7 again we get $\int_G u_k^* d u_k u_h \, \mathrm{d}\, \mu(k) = \int_G u_k^* d u_k \, \mathrm{d}\, \mu(k) u_h$, so are done. $\square$

3.3 A Result of Braga and Farah

Note that in the averaging process from the previous subsection, convergence is happening in the weak-$*$ topology of $\mathscr{B}(\mathcal{H})$. However, by Lemma 3.2.8, we know that the averaging process behaves well with uniformly finite propagation operators. In this subsection, we present a result of Braga and Farah from [5, Lemma 4.9] (see Theorem 3.3.2 below) which will allow us to work with uniformly finite propagation operators. This theorem will allow us to uniformly ϵ-r-approximate (Definition 2.1.2) $a \in \ell^\infty(\mathcal{U}, \mathscr{B}(\ell^2(X)))$ where $\mathcal{U}$ is the unitary group of $\ell^\infty(X)$. That is, given $\epsilon > 0$, there exists a *single* $r > 0$ for all $u \in \mathcal{U}$ such that $a \in \ell^\infty(\mathcal{U}, \mathscr{B}(\ell^2(X)))$ can be ϵ-r-approximated. Our argument is slightly different in that we only insist that the summations in the next definition converge in the weak operator topology whereas Braga and Farah use strong operator topology convergence in their proof.

To state the result, let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$ denote the closed unit disk in the complex plane. Let I be a countably infinite set, and let $\mathbb{D}^I$ denote as usual the space of all I-indexed tuples $\lambda := (\lambda_i)_{i \in I}$ with each $\lambda_i \in \mathbb{D}$. We fix this notation throughout this section.

Definition 3.3.1 (symmetrically summable)**.** A sequence $(a_i)_{i \in I}$ is *symmetrically summable* if for all $\lambda \in \mathbb{D}^I$, the sum $\sum_{i \in I} \lambda_i a_i$ converges in the weak operator topology to an element of $C_u^*(X)$. If (a_i) is symmetrically summable and $\lambda = (\lambda_i)$ is in $\mathbb{D}^I$, we write a_λ for the operator $\sum_{i \in I} \lambda_i a_i$.

Theorem 3.3.2. *Let (a_i) be a symmetrically summable collection of operators in $C_u^*(X)$. Then for any $\epsilon > 0$ there exists $r > 0$ such that for all $\lambda \in \mathbb{D}^I$, the operator a_λ is ϵ-r-approximated.*

The content of the result is the order of quantifiers: the point is that given an $\epsilon > 0$ there is an $r > 0$ that works for all the a_λ at once. The proof of Proposition 3.3.2 proceeds via an application of the Baire category theorem to the following sets.

Definition 3.3.3. Say (a_i) is symmetrically summable, and for any $\epsilon, r > 0$ define

$$U_{\epsilon,r} := \{\lambda \in \mathbb{D}^I \mid a_\lambda \text{ can be } \epsilon\text{-}r\text{-approximated}\}.$$

Note that the hypothesis of Theorem 3.3.2 says that for any $\epsilon > 0$,

$$\mathbb{D}^I = \bigcup_{r=1}^{\infty} U_{\epsilon,r}, \tag{6}$$

while the conclusion of Theorem 3.3.2 says that for any $\epsilon > 0$ there exists r such that $\mathbb{D}^I = U_{\epsilon,r}$.

We equip $\mathbb{D}^I$ with the product topology, which is compact (by Tychonoff's theorem), so in particular a space to which the Baire category theorem applies.

We will first show that the sets in Definition 3.3.3 are closed for any symmetrically summable (a_i). Then we will show that if (a_i) does not satisfy the conclusion of Theorem 3.3.2, there is $\epsilon > 0$ such that for all $r > 0$, $U_{r,\epsilon}$ is nowhere dense in $\mathbb{D}^I$. As we have the union in line (6), this contradicts the Baire category theorem and we will be done.

We now embark on the proof that $U_{\epsilon,r}$ is closed. We will need two preliminary lemmas.

Lemma 3.3.4. *(i) If a is a bounded operator on $\ell^2(X)$ such that for all finite rank projections p in $\ell^\infty(X)$ the product pap can be ϵ-r-approximated, then a itself can be ϵ-r-approximated.*

(ii) Say a is a bounded operator on $\ell^2(X)$ and $\epsilon, r > 0$ are such that for all $\delta > 0$, a can be $(\epsilon+\delta)$-r-approximated. Then a can be ϵ-r-approximated.

Proof. (i) Let J be the net of all finite rank projections in $\ell^\infty(X)$, equipped with the usual operator ordering. For each $p \in J$, choose $b_p \in \mathbb{C}_u^r[X]$ such that $\|pap - b_p\| \leq \epsilon$. Then the net $(b_p)_{p \in J}$ is norm bounded, so has a weak operator topology convergent subnet, say $(b_p)_{p \in J'}$, converging to some bounded operator b on $\ell^2(X)$. Note moreover that $\lim_{p \in J'} p$ equals the identity in the weak operator topology, and so $\lim_{p \in J'} pap = a$ and $\lim_{p \in J'}(pap - b_p) = a - b$ in the weak operator topology.

Now, as weak operator topology limits do not increase norms, we see that

$$\|a - b\| \leq \limsup_{p \in J'} \|pap - b_p\| \leq \epsilon.$$

Hence to complete the proof, it suffices to show that b is in fact in $\mathbb{C}_u^r[X]$. Indeed, for each $(x,y) \in X \times X$, the function taking a bounded operator c on $\ell^2(X)$ to its matrix entry c_{xy} is weak operator topology continuous. Hence, if $d(x,y) > r$ then

$$b_{xy} = \lim_{p \in J'} \big((b_p)_{xy}\big) = 0 \text{ and so } b \in \mathbb{C}_u^r[X].$$

(ii) For each n, let $b_n \in \mathbb{C}_u^r[X]$ be such that $\|a - b_n\| \leq \epsilon + 1/n$. As in the previous part, there is a subnet $(b_{n_j})_{j \in J}$ of the sequence (b_n) that converges to some $b \in \mathbb{C}_u^r[X]$ in the weak operator topology. As weak operator topology limits cannot increase norms, we see that

$$\|a - b\| \leq \limsup_{j \in J} \|a - b_{n_j}\| \leq \limsup_{j \in J} (\epsilon + 1/n_j) = \epsilon,$$

which shows that a can be ϵ-r-approximated as claimed. □

Lemma 3.3.5. *Say $(x_i)_{i \in I}$ is a collection in a Banach space such that $\sum_i \lambda_i x_i$ converges in norm for all $(\lambda_i) \in \mathbb{D}^I$. Then for any $\delta > 0$ there exists a finite subset F of I such that for all $(\lambda_i) \in \mathbb{D}^I$*

$$\left\| \sum_{i \in I \setminus F} \lambda_i x_i \right\| < \delta.$$

Proof. For notational convenience, identify I with $\mathbb{N}$, so we are just dealing with a sequence (x_n). Assume for contradiction that there exists $\delta > 0$ such that for all N there exists $(\lambda_n) \in \mathbb{D}^{\mathbb{N}}$ such that

$$\left\| \sum_{n > N} \lambda_n x_n \right\| \geq \delta.$$

We will inductively define sequences $(\lambda^{(m)})_{m=1}^{\infty}$ of points in $\mathbb{D}^{\mathbb{N}}$ and $N_1 < M_1 < N_2 < M_2 < \cdots$ of natural numbers such that for all m,

$$\left\| \sum_{n=N_m+1}^{M_m} \lambda_n^{(m)} x_n \right\| \geq \delta/2.$$

Indeed, let $m = 1$, and let N_1 and $\lambda^{(1)}$ be such that

$$\left\| \sum_{n>N_1} \lambda_n^{(1)} x_n \right\| \geq \delta.$$

As $\sum_{n>N_1} \lambda_n^{(1)} x_n$ is norm convergent, there exists $M_1 > N_1$ such that

$$\left\| \sum_{n>M_1} \lambda_n^{(1)} x_n \right\| \leq \delta/2$$

(such exists by our convergence assumption). Now, having chosen $N_1 < M_1 < N_2 < \cdots < M_m$, let us choose $N_{m+1} > M_m$ and $(\lambda)^{(m+1)}$ so that

$$\left\| \sum_{n>N_{m+1}} \lambda_n^{(m+1)} x_n \right\| \geq \delta,$$

and choose $M_{m+1} > N_{m+1}$ such that

$$\left\| \sum_{n>M_{m+1}} \lambda_n^{(m+1)} x_n \right\| \leq \delta/2.$$

Then the constructed sequences have the desired properties.

Now, define a new sequence $\lambda \in \mathbb{D}^{\mathbb{N}}$ by the formula

$$\lambda_n := \begin{cases} \lambda_n^{(m)}, & N_m < n \leq M_m \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{n=1}^{\infty} \lambda_n x_n$ converges in norm. In particular, it is Cauchy. This implies that for all suitably large m, $\| \sum_{n=N_m+1}^{M_m} \lambda_n x_n \| < \delta/2$, which contradicts the properties of our construction. $\square$

Lemma 3.3.6. *Say (a_i) is a symmetrically summable collection. Then for any $\epsilon, r > 0$ the set $U_{\epsilon,r}$ of Definition 3.3.3 is closed.*

Proof. Assume for contradiction that for some $\epsilon, r > 0$, $U_{\epsilon,r}$ is not closed. Then there exists some $\lambda \in \overline{U_{\epsilon,r}} \setminus U_{\epsilon,r}$. As $\lambda \notin U_{\epsilon,r}$, we have that a_λ cannot be ϵ-r-approximated. Using (the contrapositive of) Lemma 3.3.4, part (i), there exists a finite rank projection $p \in \ell^\infty(X)$ such that $pa_\lambda p$ cannot be ϵ-r-approximated.

Now, for any $\eta \in \mathbb{D}^I$, the sum $\sum_{i\in I} \eta_i a_i$ defining a_η is weakly convergent. As p is finite rank, this implies that the sum $\sum_{i\in I} p\eta_i a_i p$ is norm convergent (cf. 2.3.2). Hence using Lemma 3.3.5, for any $\delta > 0$ there exists a finite subset F of I such that

$$\left\| \sum_{i\in I\setminus F} p\eta_i a_i p \right\| < \delta \tag{7}$$

for all $\eta \in \mathbb{D}^I$ (and in particular for $\eta = \lambda$). As F is finite, the set

$$\left\{ \eta \in \mathbb{D}^I \;\middle|\; |F| \max_{i\in F} \|a_i\| |\eta_i - \lambda_i| < \delta \text{ for all } i \in F \right\} \tag{8}$$

is an open neighborhood of λ for the product topology. As λ is in the closure of $U_{\epsilon,r}$, the set in line (8) contains some $\theta \in U_{\epsilon,r}$. In particular $pa_\theta p$ can be ϵ-r-approximated, so there is $b \in \mathbb{C}_u^r[X]$ be such that $\|pa_\theta p - b\| \leq \epsilon$.

Note that

$$\|pa_\lambda p - b\| \leq \|pa_\theta p - b\| + \|pa_\lambda p - pa_\theta p\|$$

$$\leq \|pa_\theta p - b\| + \left\| \sum_{i\in F}(\lambda_i - \theta_i)pa_ip \right\| + \left\| \sum_{i\in I\setminus F} \theta_i pa_ip \right\| + \left\| \sum_{i\in I\setminus F} \lambda_i pa_ip \right\|.$$

The first term on the bottom line is bounded above by ϵ by choice of b, the second is bounded above by δ using that θ is in the set in line (8), and the third and fourth terms are bounded above by δ using the estimate in line (7) (which is valid for all elements η of $\mathbb{D}^I$).

Thus, we have shown that for arbitrary $\delta > 0$, there exists $b \in \mathbb{C}_u^r[X]$ such that $\|pa_\lambda p - b\| \leq \epsilon + 3\delta$. Using Lemma 3.3.4, part (ii), this implies that $pa_\lambda p$ can be ϵ-r-approximated. This contradicts our assumption in the first paragraph, so we are done. □

Now we turn to showing that if the conclusion of Theorem 3.3.2 is false, then for suitably small $\epsilon > 0$, all the sets $U_{\epsilon,r}$ of Definition 3.3.3 are nowhere dense in $\mathbb{D}^I$. We need another two preliminary lemmas.

Lemma 3.3.7. *If K is a norm-compact subset of $C_u^*(X)$ then for any $\epsilon > 0$ there exists $r > 0$ such that all operators in K can be ϵ-r-approximated.*

Proof. We choose a finite subset $\{a_1, ..., a_n\} \subseteq K$ such that every point of K is within $\epsilon/2$ of an element of $\{a_1, ..., a_n\}$. As each a_i is in $C_u^*(X)$, it can be $\epsilon/2$-r_i-approximated for some r_i. Is then straightforward to see that $r = \max\{r_1, ..., r_n\}$ has the desired property. □

Lemma 3.3.8. *Let (a_i) be a symmetrically summable collection that does not satisfy the conclusion of Proposition 3.3.2. Then there is an $\epsilon > 0$ so that for all $r > 0$ and all finite subsets F of I there exists $(\lambda_i) \in \mathbb{D}^I$ such that $\sum_{i\in I\setminus F} \lambda_i a_i$ cannot be ϵ-r approximated.*

Proof. Let (a_i) be as in the statement. Then there exists $\delta > 0$ such that for all $r > 0$ there exists $\lambda \in \mathbb{D}^I$ such that a_λ is not δ-r-approximable. Assume for contradiction that the conclusion of the lemma fails. Then there exists $s > 0$ and a finite subset F of I such that for all $(\lambda_i) \in \mathbb{D}^I$ we have that $\sum_{i \in I \setminus F} \lambda_i a_i$ is $\delta/2$-s-approximated. As F is finite, the set

$$K := \left\{ \sum_{i \in F} \lambda_i a_i \;\middle|\; \lambda \in \mathbb{D}^I \right\}$$

is norm-compact. Hence Lemma 3.3.7 gives $t > 0$ such that every element of K can be $\delta/2$-t-approximated. Now, for arbitrary $\lambda \in \mathbb{D}^I$,

$$a_\lambda = \sum_{i \in F} \lambda_i a_i + \sum_{i \in I \setminus F} \lambda_i a_i;$$

as the first term above can be $\delta/2$-s-approximated, and as the second can be $\delta/2$-t-approximated, this implies that a_λ can be δ-$\max\{s,t\}$-approximated. As λ was arbitrary, this contradicts the first sentence in the proof, and we are done. □

As already noted after the statement of Theorem 3.3.2, the following lemma completes the proof of the theorem.

Lemma 3.3.9. *Say (a_i) is a symmetrically summable collection that does not satisfy the conclusion of Proposition 3.3.2. Then there is $\epsilon > 0$ such that for each $r > 0$ the set $U_{\epsilon,r}$ of Definition 3.3.3 is nowhere dense in $\mathbb{D}^I$.*

Proof. Let $\epsilon' > 0$ have the property from Lemma 3.3.8. We claim that $\epsilon := \epsilon'/2$ has the property required for this lemma. Assume for contradiction that for some $r > 0$, $U_{\epsilon,r}$ is not nowhere dense. Lemma 3.3.6 implies that $U_{\epsilon,r}$ is closed, and so it contains a point λ in its interior. Then by definition of the product topology there exists a finite set $F \subseteq I$ and $\delta > 0$ such that the set

$$V := \{\eta \in \mathbb{D}^I \mid |\eta_i - \lambda_i| < \delta \text{ for all } i \in F\} \tag{9}$$

is contained in $U_{\epsilon,r}$.

Note that the element $\sum_{i\in F}\lambda_i a_i$ is in $C_u^*(X)$ by assumption, so can be ϵ-s-approximated for some s; let $b_\lambda \in \mathbb{C}_u^s[X]$ be such that $\|\sum_{i\in F}\lambda_i a_i - b_\lambda\| \leq \epsilon$. On the other hand, Lemma 3.3.8 gives us $\eta \in \mathbb{D}^I$ so that $\sum_{i\in I\setminus F}\eta_i a_i$ cannot be ϵ'-$\max\{r,s\}$-approximated. We may further assume that $\eta_i = 0$ for $i \in F$. Define $\theta \in \mathbb{D}^I$ by

$$\theta_i := \begin{cases} \lambda_i & i \in F \\ \eta_i & i \notin F \end{cases}$$

Then θ is clearly in the set V of line (9), and so a_θ is ϵ-r-approximated. Let then $b_\theta \in \mathbb{C}_u^r[X]$ be such that $\|a_\theta - b_\theta\| \leq \epsilon$. We then see that

$$\|a_\eta - (b_\theta - b_\lambda)\| \leq \|a_\eta - a_\theta + b_\lambda\| + \|a_\theta - b_\theta\| \leq \left\| b_\lambda - \sum_{i\in F}\lambda_i a_i \right\| + \|a_\theta - b_\theta\|$$

The terms on the right are each less than ϵ by choice of b_λ and b_θ, and so $\|a_\eta - (b_\theta - b_\lambda)\| \leq 2\epsilon = \epsilon'$. As $b_\lambda + b_\theta$ has propagation at most $\max\{r,s\}$, this contradicts the assumption that a_η cannot be ϵ'-$\max\{r,s\}$-approximated, so we are done. $\square$

3.4 All Bounded Derivations on Uniform Roe Algebras are Inner

Proof of Theorem 3.0.1. Let $\delta : C_u^*(X) \to C_u^*(X)$ be a derivation. Theorem 3.1.3 implies that δ is spatially implemented, so there is $d \in \mathscr{B}(\ell^2(X))$ such that $\delta(a) = [a,d]$ for all $a \in C_u^*(X)$. We will show that d is in $C_u^*(X)$.

Let $\mathcal{U}$ be the unitary group of $\ell^\infty(X)$, equipped with the discrete topology. As $\mathcal{U}$ is abelian, it is amenable (see for example [3, Theorem G.2.1]), and so we may fix a right-invariant mean on $\ell^\infty(\mathcal{U})$. As in Lemma 3.2.1 above, this allows us to build a right-invariant, contractive, linear map

$$\ell^\infty(\mathcal{U}, \mathscr{B}(\ell^2(X))) \to \mathscr{B}(\ell^2(X)), \quad a \mapsto \int_{\mathcal{U}} a(u)\,\mathrm{d}\,\mu(u). \tag{10}$$

We apply this to the bounded function

$$\mathcal{U} \to \mathscr{B}(\ell^2(X)), \quad u \mapsto u^* du$$

to get a bounded operator

$$d' := \int_{\mathcal{U}} u^* du \,\mathrm{d}\,\mu(u) \in \mathscr{B}(\ell^2(X)).$$

Using Lemma 3.2.9 applied to the identity representation of $\mathcal{U}$, d' is in the commutant of $\mathcal{U}$. As $\mathcal{U}$ spans $\ell^\infty(X)$, and as $\ell^\infty(X)$ is maximal abelian in $\mathscr{B}(\ell^2(X))$, this implies that d' is in $\ell^\infty(X)$. To show that d is in $C_u^*(X)$, it therefore suffices to show that $h := d - d'$ is in $C_u^*(X)$.

Continuing, let $p_x \in \mathscr{B}(\ell^2(X))$ be the rank one projection onto the span of the Dirac mass at x. For an element f of the unit ball of $\ell^\infty(X)$ (considered as a multiplication operator on $\ell^2(X)$), write f as a strongly convergent sum

$$f = \sum_{x \in X} f(x) p_x.$$

Then using strong continuity of subtraction, and separate strong continuity of multiplication on bounded sets,

$$[f, d] = \Big[\sum_{x \in X} f(x) p_x, d \Big] = \sum_{x \in X} f(x)[p_x, d].$$

On the other hand, by the assumption that δ is a derivation on $C_u^*(X)$, $[f, d]$ is in $C_u^*(X)$ for all $f \in \ell^\infty(X)$. It follows that if we set $I = X$, and if for each $x \in X$ we set $a_x := [p_x, d]$, then the collection $(a_x)_{x \in X}$ satisfies the assumptions of Proposition 3.3.2. Hence, for every $\epsilon > 0$ there exists $r > 0$ such that for every f in the unit ball of $\ell^\infty(X)$, the operator $[f, d]$ can be ϵ-r-approximated. In particular, using that any $u \in \mathcal{U}$ has propagation zero and norm one, for any $\epsilon > 0$ there exists $r > 0$ such that $d - u^* du = u^*[u, d]$ can be ϵ-r-approximated.

For each $u \in \mathcal{U}$, we can therefore choose $a(u)$ of propagation at most r such that $b(u) := d - u^*du - a(u)$ has norm at most ϵ. Note that the functions $a : u \mapsto a(u)$ and $b : u \mapsto b(u)$ are in $\ell^\infty(\mathcal{U}, \mathscr{B}(\ell^2(X)))$. Hence we may consider their images under the map in line (10).

Using that the map in line (10) is linear and acts as the identity on constant functions (see Lemma 3.2.1), we see that

$$\begin{aligned}\int_{\mathcal{U}} a(u)\,\mathrm{d}\,\mu(u) + \int_{\mathcal{U}} b(u)\,\mathrm{d}\,\mu(u) &= \int_{\mathcal{U}} d - u^*du\,\mathrm{d}\,\mu(u) = d - \int_{\mathcal{U}} u^*du\,\mathrm{d}\,\mu(u) \\ &= d - d' = h. \end{aligned} \tag{11}$$

On the other hand, $\int_{\mathcal{U}} a(u)\,\mathrm{d}\,\mu(u)$ has propagation at most r by Lemma 3.2.8, and $\int_{\mathcal{U}} b(u)\,\mathrm{d}\,\mu(u)$ has norm at most ϵ as the map in line (10) is contractive (see Lemma 3.2.1). In particular, line (11) writes h as a sum of an element of $C^*_u(X)$, and an element of norm at most ϵ. As ϵ was arbitrary, h is in $C^*_u(X)$, and we are done. $\square$

4 Hochschild Cohomology

To motivate this section first note that the Hochschild coboundary operator from a C*-algebra $\mathcal{A}$ to the linear maps from $\mathcal{A}$ to itself is given by

$$\partial a(b) = ab - ba, \ a, b \in \mathcal{A}.$$

Thus, ∂a is an inner derivation. Next, the coboundary operator from linear maps to bilinear maps from $\mathcal{A}$ to itself is given by

$$\partial\phi(a, b) = a\phi(b) - \phi(ab) + \phi(a)b.$$

Hence, the kernel of this coboundary operator is the set of derivations on $\mathcal{A}$. So, taking this kernel and modding out by the image of the previous coboundary, if zero, means that all derivations on $\mathcal{A}$ are inner. Thus, by the previous section, the first Hochschild cohomology of uniform Roe algebras associated to bounded geometry metric spaces vanishes. This naturally leads to the question of wether or not the higher dimensional Hochschild cohomology groups of the uniform Roe algebra vanish also.

In this section we introduce Hochschild cohomology, its construction, and several properties of Hochschild cohomology.

4.1 Multilinear Maps

In this subsection we recall some properties of multilinear maps that we will need.

Definition 4.1.1 (Multilinear map)**.** Let $\mathcal{A}$ and $\mathcal{V}$ be vector spaces. An n-linear map, or a *multilinear* map, is a map from the n-fold product of $\mathcal{A}$ to $\mathcal{V}$,

$$\phi : \mathcal{A}^n \to \mathcal{V},$$

that is separately linear in each of its variables. That is, for an arbitrary

$i \in \{1, \ldots, n\}$, holding the other coordinates steady, and $x, y \in \mathcal{A}$, $\lambda \in \mathbb{C}$ we have

$$\phi\Big(a_1, \ldots, a_{i-1}, \lambda x + y, a_{i+1}, \ldots, a_n\Big)$$
$$= \lambda\phi\Big(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n\Big)$$
$$+\phi\Big(a_1, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_n\Big).$$

Lemma 4.1.2. *Let $\phi : \mathcal{A}^n \to \mathcal{V}$ be a multilinear map between Banach spaces. Then ϕ is separately norm continuous if and only if there exists an $M > 0$ such that*

$$\|\phi(a_1, \ldots, a_n)\|_{\mathcal{V}} \leq M \|a_1\|_{\mathcal{A}} \cdots \|a_n\|_{\mathcal{A}}.$$

When such an M exists we say that ϕ is bounded.

Proof. We induct on n. If $n = 1$ this is just the standard result about linear transformations between normed spaces, see for example [6] Proposition III.2.1. Next, suppose our result holds for all k such that $1 \leq k \leq n-1$; we show that it holds for n.

First, suppose that ϕ is separately norm continuous. For each $a := (a_1, \ldots, a_{n-1}) \in \mathcal{A}^{n-1}$ define

$$\phi_a : \mathcal{A} \to \mathcal{V} \text{ by } x \mapsto \phi(a_1, \ldots, a_{n-1}, x).$$

Additionally, for each fixed $x \in \mathcal{A}$ define $\phi_x : \mathcal{A}^{n-1} \to \mathcal{V}$

$$\text{by } (a_1, \ldots, a_{n-1}) \mapsto \phi(a_1, \ldots, a_{n-1}, x).$$

Note that since ϕ is separately norm continuous both ϕ_a and ϕ_x are bounded by the inductive hypothesis. Thus, for each $x \in \mathcal{A}$, there exists an $M_x > 0$ such that

$$\sup_{\|a_1\|, \ldots, \|a_{n-1}\|=1} \|\phi_a(x)\| = \sup_{\|a_1\|, \ldots, \|a_{n-1}\|=1} \|\phi_x(a)\|$$
$$\leq \sup_{\|a_1\|, \ldots, \|a_{n-1}\|=1} M_x \|a_1\| \cdots \|a_{n-1}\| = M_x < \infty.$$

By the uniform boundedness principle this implies that there exists an $M > 0$ such that

$$\sup_{\|a_1\|,\ldots,\|a_{n-1}\|=1} \|\phi_a(x)\| \leq M \|x\| \text{ for all } x \in \mathcal{A}.$$

Note that, since ϕ is multilinear, if any of the $a_k \in \{a_1, \ldots, a_{n-1}\}$ is zero then

$$\phi(a_1, \ldots, a_{n-1}, x) = 0.$$

Now suppose that $a_k \neq 0$ for all $a_k \in \{a_1, \ldots, a_{n-1}\}$. Observe that

$$\frac{1}{\|a_1\| \cdots \|a_{n-1}\|} \|\phi(a_1, \ldots, a_{n-1}, x)\| \leq \sup_{\|a_1\|,\ldots,\|a_{n-1}\|=1} \|\phi_a(x)\| \leq M \|x\|.$$

Rearranging, we have that

$$\|\phi(a_1, \ldots, a_{n-1}, x)\| \leq M \|x\| \|a_1\| \cdots \|a_{n-1}\|$$

and so ϕ is bounded.

Next, suppose that ϕ is bounded. Thus, there exists an $M > 0$ such that

$$\|\phi(a_1, \ldots, a_n)\| \leq M \|a_1\| \cdots \|a_n\|.$$

Let $a_1, \ldots, a_{n-1} \in \mathcal{A}$ and let $\epsilon > 0$ be given. Set $\delta = \frac{\epsilon}{M\|a_1\|\cdots\|a_{n-1}\|}$ and let $x, y \in \mathcal{A}$ satisfy $\|x - y\| < \delta$. Then,

$$\|\phi(a_1, \ldots, a_{n-1}, x - y)\| \leq M \|x - y\| \|a_1\| \cdots \|a_{n-1}\| < \epsilon$$

so that ϕ is separately norm continuous. □

Definition 4.1.3 (operator norm)**.** Let $\mathcal{A}$ and $\mathcal{V}$ be Banach spaces and let $\phi : \mathcal{A}^n \to \mathcal{V}$ be a bounded multilinear map. We define the operator norm of ϕ by

$$\|\phi\|_{op} = \sup_{\|a_1\|,\ldots,\|a_n\|\leq 1} \|\phi(a_1, \ldots, a_n)\|_{\mathcal{V}}.$$

Lemma 4.1.4. *Let $\mathcal{A}$ be a C^*-subalgebra of a von Neumann algebra $\mathcal{M}$. Additionally, let $\mathcal{V}$ be the dual space of a Banach space $\mathcal{V}_*$. If $\overline{\mathcal{A}}$ is the natural ultraweak closure inherited from $\mathcal{M}$ and $\phi : (\overline{\mathcal{A}})^n \to \mathcal{V}$ is a multilinear map that is separately ultraweak-weak* continuous, then, ϕ is bounded (and so separately norm continuous).*

Proof. By Lemma 4.1.2 , it suffices to consider $n = 1$. Suppose that $\phi : \overline{\mathcal{A}} \to \mathcal{V}$ is an ultraweak-weak* continuous map. Now, suppose for contradiction that ϕ is not bounded. Then for each $n \in \mathbb{N}$ there exists an $x_n \in \overline{\mathcal{A}}$ such that $\|\phi(x_n)\| > n\|x_n\|$. Since the inequality is strict and our map is linear, $x_n \neq 0$ for all n so we may set $z_n = \frac{x_n}{\|x_n\|}$ so that

$$\|\phi(z_n)\| > n \quad \text{and} \quad \{z_n\}_{n\in\mathbb{N}} \subseteq \left\{a \in \overline{\mathcal{A}} : \|a\| = 1\right\}.$$

Note that, this sequence is contained is the closed unit ball of $\overline{\mathcal{A}}$ which is ultraweakly compact by the Banach-Alaoglu theorem. Thus, it contains a convergent subnet, say $\{z_k\}_{k\in K}$, converging ultraweakly to some $z \in \overline{\mathcal{A}}$. Let $h : K \to \mathbb{N}$ be the monotone final function of this subnet. Next, for $w \in \mathcal{V}_*$ and $v \in \mathcal{V}$ let $\langle w, v\rangle$ be the pairing between w and v. Then, since ϕ is ultraweak-weak* continuous, $\phi(z_k) \to \phi(z)$ weak*. Thus, given $w \in \mathcal{V}_*$ there exists a $k_0 \in K$ such that

$$|\langle w, \phi(z_k - z)\rangle| \leq \|w\| \quad \text{whenever} \quad k \geq k_0$$

so that

$$\sup_{k\geq k_0} |\langle w, \phi(z_k)\rangle| \leq \|w\| + \|\phi(z)\| \, \|w\| = (1 + \|\phi(z)\|) \, \|w\| \, .$$

Hence, by the uniform boundedness principle we have that [1]

$$\sup_{k\geq k_0} \|\phi(z_k)\| \leq M < \infty \tag{12}$$

[1]Updated 9/28/21

Next, let $n > M$. Then there exists a $k_1 \in K$ such that $h(k_1) > n$. Additionally, there exists a $k_2 \geq k_1, k_0$ so that $h(k_2) > n$. Thus, $\|\phi(z_{k_2})\| \geq n$ which contradicts (12). $\square$

4.2 The Hochschild Complex

In this subsection we discuss the Hochschild complex and define its cohomology.

Definition 4.2.1 (Dual module)**.** Let $\mathcal{A}$ be a C*-algebra. we say that $\mathcal{V}$ is a *dual module* of $\mathcal{A}$ if:

(i) $\mathcal{V}$ is a Banach $\mathcal{A}$-bimodule (Definition 3.2.3),

(ii) $\mathcal{V}$ has a pre-dual $\mathcal{V}_*$,

(iii) and the maps

$$\mathcal{V} \to \mathcal{V} \text{ defined by } x \mapsto ax \text{ and } x \mapsto xa$$

are weak* continuous for all $a \in \mathcal{A}$.

Definition 4.2.2 (Dual normal module)**.** Let $\mathcal{M}$ be a von Neumann algebra. we say that $\mathcal{V}$ is a *dual normal module* of $\mathcal{M}$ if:

(i) $\mathcal{V}$ is a dual $\mathcal{M}$-bimodule,

(ii) and the maps

$$\mathcal{M} \to \mathcal{V} \text{ defined by } m \mapsto mx \text{ and } m \mapsto xm$$

are ultraweak - weak* continuous for all $x \in \mathcal{V}$.

Definition 4.2.3 (Subdual)**.** Let $\mathcal{A}$ be a C*-algebra and let $\mathcal{V}$ be an $\mathcal{A}$-submodule of a dual module (as in Definition 4.2.1) $\mathcal{W}$ (under the same action) that is also a dual space; that is, $\mathcal{W}$ has a predual $\mathcal{W}_*$ where $(\mathcal{W}_*)^* \cong \mathcal{W}$. We will call such a module $\mathcal{V}$ a *subdual* of $\mathcal{W}$. Note that we are not requiring $\mathcal{V}$ to be a dual space, just that it is a submodule of a specified dual space. Moreover, if $\mathcal{A}$ is a C*-subalgebra of a von Neumann algebra $\mathcal{M}$ where $\mathcal{W}$ is a dual normal $\mathcal{M}$-module and the action of $\mathcal{A}$ on $\mathcal{V}$ is the inherited $\mathcal{M}$-action on $\mathcal{W}$ then we say that $\mathcal{V}$ is a *subdual normal* $\mathcal{A}$-module of $\mathcal{W}$.

An example of a subdual normal module is the uniform Roe algebra acting on itself by multiplication. $C_u^*(X)$ acts on $\mathscr{B}(\ell^2(X))$ by multiplication making $\mathscr{B}(\ell^2(X))$ a $C_u^*(X)$-module. $\mathscr{B}(\ell^2(X))$ is a dual space with predual $\mathcal{L}^1(\ell^2(X))$, the trace class operators. So $C_u^*(X)$ is a submodule of the dual space $\mathscr{B}(\ell^2(X))$. However, $C_u^*(X)$ is not usually a dual space. This additional structure on the submodule allows us to use the relative weak* topology inherited from the parent module.

By $\mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$ we mean the vector space of separately norm continuous multilinear maps from the n-fold Cartesian product of $\mathcal{A}$ to the $\mathcal{A}$-bimodule $\mathcal{V}$ when $n \geq 1$ and $\mathscr{L}_c^0(\mathcal{A}, \mathcal{V}) := \mathcal{V}$.

Let $\mathcal{A}$ be a concrete C*-algebra. If $\mathcal{W}$ is a dual normal $\mathcal{A}$-bimodule with subdual $\mathcal{V}$, we use the notation $\mathscr{L}_w^n(\mathcal{A}, \mathcal{V})$ to indicate the vector space of multilinear maps that are separately ultraweak-weak* continuous; that is, for $\phi \in \mathscr{L}_w^n(\mathcal{A}, \mathcal{V})$

$$\text{if } \{a_\alpha\} \subset \mathcal{A} \text{ is a net such that } a_\alpha \to a \in \mathcal{A} \text{ ultraweakly in } \mathscr{B}(\mathcal{H})$$

$$\text{then } \phi(\ldots, a_\alpha, \ldots) \to \phi(\ldots, a, \ldots) \in \mathcal{V} \text{ weak* in } \mathcal{W}.$$

When we write $\mathscr{L}^n(\mathcal{A},\mathcal{V})$ then either subscript may be attached. Considering $\mathcal{A}$ as a module over itself we will simply write $\mathscr{L}(\mathcal{A})$. Additionally, we equip both $\mathscr{L}^n_c(\mathcal{A},\mathcal{V})$ and $\mathscr{L}^n_w(\mathcal{A},\mathcal{V})$ with the operator norm.

Remark 4.2.4. Note that while $\mathscr{L}^n_c(\mathcal{A},\mathcal{V})$ is complete in norm, we are not assuming, nor do we require these vector spaces to be complete in norm.

To define the Hochschild cohomology we first construct the cochain complex

$$0 \to \mathscr{L}^0(\mathcal{A},\mathcal{V}) \xrightarrow{\partial} \mathscr{L}^1(\mathcal{A},\mathcal{V}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathscr{L}^n(\mathcal{A},\mathcal{V}) \xrightarrow{\partial} \mathscr{L}^{n+1}(\mathcal{A},\mathcal{V}) \xrightarrow{\partial} \cdots$$

for both the norm continuous and ultraweak-weak* continuous cases where the coboundary operator $\partial : \mathscr{L}^n(\mathcal{A},\mathcal{V}) \to \mathscr{L}^{n+1}(\mathcal{A},\mathcal{V})$ is defined by

$$\begin{aligned}(\partial\phi)(a_1,\ldots,a_{n+1}) &= a_1\phi(a_2,\ldots,a_{n+1})\\ &+\sum_{j=1}^{n}(-1)^j\phi(a_1,\ldots,a_ja_{j+1},\ldots,a_{n+1})\\ &+(-1)^{n+1}\phi(a_1,\ldots,a_n)a_{n+1} \quad (n\geq 1)\end{aligned}$$

and for $n = 0$

$$(\partial v)(a) = av - va \quad (v \in \mathcal{V}, a \in \mathcal{A}).$$

A straightforward calculation shows that ∂^2 is always zero. The n^{th} Hochschild cohomology group $H^n_c(\mathcal{A},\mathcal{V})$ (resp. $H^n_w(\mathcal{A},\mathcal{V})$ in the ultraweak-weak* case) is the quotient vector space

$$H^n(\mathcal{A},\mathcal{V}) := \frac{\ker(\partial : \mathscr{L}^n(\mathcal{A},\mathcal{V}) \to \mathscr{L}^{n+1}(\mathcal{A},\mathcal{V}))}{\operatorname{im}(\partial : \mathscr{L}^{n-1}(\mathcal{A},\mathcal{V}) \to \mathscr{L}^n(\mathcal{A},\mathcal{V}))}.$$

Additionally, when we consider $\mathcal{A}$ as a module over itself we simply write $H^n(\mathcal{A})$. The cohomology obtained from this construction is the *Hochschild* cohomology. We call an element $\phi \in \ker(\partial : \mathscr{L}^n(\mathcal{A}, \mathcal{V}) \to \mathscr{L}^{n+1}(\mathcal{A}, \mathcal{V}))$ a *cocycle*, and we call an element $\psi \in \operatorname{im}(\partial : \mathscr{L}^{n-1}(\mathcal{A}, \mathcal{V}) \to \mathscr{L}^n(\mathcal{A}, \mathcal{V}))$ a *coboundary.*

Definition 4.2.5 (multimodular maps)**.** Let $\mathcal{A}$ be a C*-algebra and let $\phi : \mathcal{A}^n \to \mathcal{V}$ be a bounded multilinear map to the Banach $\mathcal{A}$-bimodule $\mathcal{V}$. If $\mathcal{B}$ is a C*-subalgebra of $\mathcal{A}$ we say that ϕ is $\mathcal{B}$-*multimodular* if for any $b \in \mathcal{B}$ the following hold.

1. $b\phi(a_1, \ldots, a_n) = \phi(ba_1, \ldots, a_n)$,
2. $\phi(a_1, \ldots, a_{j-1}b, a_j, \ldots, a_n) = \phi(a_1, \ldots, a_{j-1}, ba_j, \ldots, a_n)$ and
3. $\phi(a_1, \ldots, a_n b) = \phi(a_1, \ldots, a_n)b$

If $\mathcal{B}$ is a C*-subalgebra of $\mathcal{A}$ we use the notation $\mathscr{L}^n(\mathcal{A}, \mathcal{V} : \mathcal{B})$ to indicate that the maps are $\mathcal{B}$-multimodular where we may use either subscript,"c" or "w". As before we may construct the Hochschild cohomology of $\mathcal{B}$-multimodular maps which we denote by $H^n(\mathcal{A}, \mathcal{V} : \mathcal{B})$ where either subscript c or w may be attached. Additionally, if we are considering $\mathcal{A}$ as a module over itself we simply write $H^n(\mathcal{A} : \mathcal{B})$.

4.3 Sinclair and Smith's 'Reduction of Cocycles'

In this subsection we introduce a method to modify a cocycle, say $\phi \in \mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$, by a coboundary to obtain an operator in $\mathscr{L}_c^n(\mathcal{A}, \mathcal{V} : \mathcal{B})$ where $\mathcal{A} \subseteq \mathscr{B}(\mathcal{H})$ is a C*-algebra, $\mathcal{B}$ is a C*-subalgebra of $\mathcal{A}$, and $\mathcal{V}$ is a dual normal $\mathscr{B}(\mathcal{H})$-bimodule.

On several occasions, due to the boundary operator and the averaging operator (which we introduce below), we will have to track the entries of the original input vector and the current coordinate position that the entry is now in after applying one of the aforementioned operators. To accomplish this we will adopt the notation $\phi(\dots,(a_j)_k,\dots)$ where "j" was the position of the entry in the original input vector and "k" is the entry's current position in ϕ.

Lemma 4.3.1 ([19] Lemma 3.2.1)**.** *Let $\mathcal{B}$ be a unital subalgebra of a unital C^*-algebra $\mathcal{A}$. Let $\mathcal{V}$ be a Banach $\mathcal{A}$-bimodule, and let $\phi \in \mathscr{L}^n(\mathcal{A},\mathcal{V})$ with $\partial\phi = 0$.*

Then for all $b \in \mathcal{B}$ and $x_1,\dots,x_n \in \mathcal{A}$ we have:

(i) $\phi(b,x_2,\dots,x_n) = 0$ *if and only if*
$\phi(1,x_2,\dots,x_n) = 0$ *and* $\phi(bx_1,x_2,\dots,x_n) = b\phi(x_1,\dots,x_n)$.

(ii) *Fix $k \le n$. Then for all $j \in \{2,\dots,k\}$,*
$\phi(x_1,\dots,x_{j-1},b,x_{j+1},\dots,x_n) = 0$ *if and only if*
$\phi(x_1,\dots,x_{j-1},1,x_{j+1},\dots,x_n) = 0$ *and*
$\phi(x_1,\dots,x_{j-1}b,x_j,\dots,x_n) = \phi(x_1,\dots,x_{j-1},bx_j,\dots,x_n)$

(iii) *Additionally,*
$\phi(x_1,\dots,x_{n-1},b) = 0$ *if and only if*
$\phi(x_1,\dots,x_{n-1},1) = 0$ *and* $\phi(x_1,\dots,x_nb) = \phi(x_1,\dots,x_n)b$

Proof. (i) Note that the reverse direction is immediate. Now suppose that $\phi(b,x_2,\dots,x_n) = 0$. Since ϕ is a cocycle we have that

$$0 = \partial\phi(b,x_1,\dots,x_n) = b\phi(x_1,\dots,x_n) - \phi(bx_1,\dots,x_n)$$

$$+\sum_{i=1}^{n-1}(-1)^{i+1}\phi(b,x_1,\dots,x_ix_{i+1},\dots,x_n) + (-1)^{n+1}\phi(b,x_1,\dots,x_{n-1})x_n$$

By supposition the last line is zero. Thus, $b\phi(x_1,\dots,x_n) = \phi(bx_1,\dots,x_n)$.

(ii) Note that the reverse direction is immediate. Now suppose that $\phi(x_1, \ldots, x_{j-1}, b, x_{j+1}, \ldots, x_n) = 0$ whenever $1 < j \leq k$. Observe that

$$0 = \partial\phi\Big((x_1)_1, \ldots, (x_{j-1})_{j-1}, (b)_j, (x_j)_{j+1}, \ldots, (x_n)_{n+1}\Big)$$

$$= x_1\phi\Big((x_2)_1, \ldots, (x_{j-1})_{j-2}, (b)_{j-1}, (x_j)_j, \ldots, (x_n)_n\Big)$$

$$+ \sum_{i=1}^{j-2}(-1)^i\phi\Big((x_1)_1, \ldots, (x_ix_{i+1})_i, \ldots, (x_{j-1})_{j-2}, (b)_{j-1}, (x_j)_j, \ldots, (x_n)_n\Big)$$

$$+ (-1)^{j-1}\phi(x_1, \ldots, x_{j-1}b, x_j, \ldots, x_n) \tag{13}$$

$$+ (-1)^j\phi(x_1, \ldots, x_{j-1}, bx_j, \ldots, x_n) \tag{14}$$

$$+ \sum_{i=j}^{n-1}(-1)^{i+1}\phi\Big((x_1)_1, \ldots, (x_{j-1})_{j-1}, (b)_j, (x_j)_{j+1}, \ldots, (x_ix_{i+1})_{i+1}, \ldots, (x_n)_n\Big)$$

$$+(-1)^{n+1}\phi\Big((x_1)_1, \ldots, (x_{j-1})_{j-1}, (b)_j, (x_j)_{j+1}, \ldots, (x_{n-1})_n\Big)x_n.$$

By supposition (13) and (14) are the only nonzero lines and are of opposite sign. Hence, $\phi(x_1, \ldots, x_{j-1}b, x_j, \ldots, x_n) = \phi(x_1, \ldots, x_{j-1}, bx_j, \ldots, x_n)$.

(iii) The reverse direction is immediate. Suppose that $\phi(x_1, \ldots, x_{n-1}, b) = 0$. Then

$$0 = \partial\phi(x_1, \ldots, x_n, b)$$

$$= x_1\phi(x_2, \ldots, x_n, b) + \sum_{i=1}^{n-1}(-1)^i\phi(x_1, \ldots, x_ix_{i+1}, \ldots, b)$$

$$+(-1)^n\phi(x_1, \ldots, x_nb) + (-1)^{n+1}\phi(x_1, \ldots, x_n)b.$$

Note that the last line is the only nonzero line and its terms are of opposite sign. Thus, $\phi(x_1, \ldots, x_nb) = \phi(x_1, \ldots, x_n)b$. $\square$

Lemma 4.3.2 ([19] Lemma 3.2.4)**.** *Let $\mathcal{B}$ be the C^*-algebra spanned by an amenable group $\mathcal{U}$ of unitaries in a unital C^*-subalgebra $\mathcal{A}$, and let $\mathcal{W}$ be a dual Banach $\mathcal{A}$-bimodule. There is a continuous linear map*

$$K_n : \mathscr{L}_c^n(\mathcal{A}, \mathcal{W}) \to \mathscr{L}_c^{n-1}(\mathcal{A}, \mathcal{W}) \ \ (\textit{depending on } \mathcal{U})$$

such that if $\phi \in \mathscr{L}_c^n(\mathcal{A}, \mathcal{W})$ satisfies $\partial\phi = 0$ then $\phi - \partial(K_n\phi)$ is $\mathcal{B}$-multimodular. Moreover, we have that

$$\|K_n\| \leq \tfrac{(n+2)^n - 1}{n+1}.$$

We will prove this lemma inductively using three claims after defining the map K_n.

The map K_n is constructed recursively via

$$J_1 : \mathscr{L}_c^n(\mathcal{A}, \mathcal{W}) \to \mathscr{L}_c^{n-1}(\mathcal{A}, \mathcal{W}) \ \ \text{defined by}$$

$$(J_1\phi)(a_1, \ldots, a_{n-1}) = \int_{\mathcal{U}} u^*\phi(u, a_1, \ldots, a_{n-1})\,\mathrm{d}\,\mu(u), \tag{15}$$

$$G_k : \mathscr{L}_c^n(\mathcal{A}, \mathcal{W}) \to \mathscr{L}_c^{n-1}(\mathcal{A}, \mathcal{W}) \ \ \text{defined by}$$

$$(G_k\phi)(a_1, \ldots, a_{n-1}) = \int_{\mathcal{U}} \phi(a_1, \ldots, a_k u^*, u, a_{k+1}, \ldots, a_{n-1})\,\mathrm{d}\,\mu(u), \tag{16}$$

$$J_{k+1} : \mathscr{L}_c^n(\mathcal{A}, \mathcal{W}) \to \mathscr{L}_c^{n-1}(\mathcal{A}, \mathcal{W}) \ \ \text{defined by} \ \ J_{k+1} = J_k + (-1)^k G_k(I - \partial J_k),$$

$$\text{and } K_n = J_n. \tag{17}$$

For the first claim we will show that $\|J_1\| = 1$ and for a cocycle $\phi \in \mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$ the map $(\phi - \partial J_1\phi)(a_1, \ldots, a_n) = 0$ if $a_1 \in \mathcal{B}$. This will be our base case.

In the next claim we assume our map has been constructed so that $(\phi - \partial J_k\phi)(a_1, \ldots, a_n) = 0$ if $a_j \in \mathcal{B},\ j \in \{1, \ldots, k\}$ and that

$$\|J_k\| \leq \frac{(n+2)^k - 1}{n+1}.$$

Then we show that $(\phi - \partial J_{k+1}\phi)(a_1, \ldots, a_n) = 0$ if $a_j \in \mathcal{B}$, for some $j \in \{1, \ldots, k\}$.

Lastly, to complete the induction, we show that $(\phi - \partial J_{k+1}\phi)(a_1, \ldots, a_n) = 0$ if $a_{k+1} \in \mathcal{B}$ and that

$$\|J_{k+1}\| \leq 1 + (n+2)\|J_k\| \leq \frac{(n+2)^{k+1} - 1}{n+1}.$$

This will complete the proof of the lemma since, by Lemma 4.3.1, if $k = n$ then $(\phi - \partial K_n\phi)$ is $\mathcal{B}$-multimodular.

Claim 4.3.3. *Let $\mathcal{B}$ be a C^*-subalgebra spanned by an amenable group $\mathcal{U}$ of unitaries in a unital C^*-algebra $\mathcal{A}$, and let $\mathcal{W}$ be a dual Banach $\mathcal{A}$-bimodule. Then the map J_1 is continuous with $\|J_1\| \leq 1$. Additionally, if $\partial\phi = 0$ we have that $(\phi - \partial J_1\phi)(a_1, \ldots, a_n) = 0$ whenever $a_1 \in \mathcal{B}$.*

Proof. Note that the map J_1 is linear since our averaging operator is. Moreover, since our averaging operator is contractive we have

$$\|J_1\phi(a_1, \ldots, a_{n-1})\| = \left\| \int_G u^*\phi(u, a_1, \ldots, a_{n-1}) \, \mathrm{d}\,\mu(u) \right\|$$

$$\leq \sup_{u \in \mathcal{U}} \|\phi(u, a_1, \ldots, a_{n-1})\| \leq \|\phi\| \, \|a_1\| \cdots \|a_{n-1}\|.$$

Thus, J_1 is bounded with $\|J_1\| \leq 1$ and so is continuous. Next, observe that

$$\partial\phi(u, a_1, \ldots, a_n)$$

$$= \Big(u\phi(a_1, \ldots, a_n) - \phi(ua_1, \ldots, a_n) \Big)$$

$$+ \sum_{j=1}^{n-1} (-1)^{j+1} \phi(u, a_1, \ldots, a_j a_{j+1}, \ldots, a_n) + (-1)^{n+1}\phi(u, a_1, \ldots, a_{n-1})a_n = 0$$

since ϕ is a cocycle. Next,

$$a_1 u^*\phi(u, a_2, \ldots, a_n) - \Big(u^*\phi(ua_1, \ldots, a_n) - \phi(a_1, \ldots, a_n) \Big) \tag{18}$$

$$= a_1 u^* \phi(u, a_2, \ldots, a_n) - \Big(u^* \phi(ua_1, \ldots, a_n) - \phi(a_1, \ldots, a_n) \Big)$$

$$-u^* \partial \phi(u, a_1, \ldots, a_n)$$

$$= a_1 u^* \phi(u, a_2, \ldots, a_n) - \cancel{\Big(u^* \phi(ua_1, \ldots, a_n) - \phi(a_1, \ldots, a_n) \Big)}$$

$$+ \cancel{\Big(u^* \phi(ua_1, \ldots, a_n) - u^* u \phi(a_1, \ldots, a_n) \Big)}$$

$$+ \sum_{j=1}^{n-1} (-1)^j u^* \phi(u, a_1, \ldots, a_j a_{j+1}, \ldots, a_n)$$

$$+ (-1)^n u^* \phi(u, a_1, \ldots, a_{n-1}) a_n.$$

Then, by the invariance of the averaging operator, if $a_1 \in \mathcal{U}$ we have

$$\phi(a_1, \ldots, a_n)$$

$$= \phi(a_1, \ldots, a_n) + a_1 \int_{\mathcal{U}} u^* \phi(u, a_2, \ldots, a_n) \,\mathrm{d}\,\mu(u) - a_1 \int_{\mathcal{U}} (ua_1)^* \phi((ua_1), a_2, \ldots, a_n) \,\mathrm{d}\,\mu(u)$$

$$= \int_{\mathcal{U}} \phi(a_1, \ldots, a_n) + a_1 u^* \phi(u, a_2, \ldots, a_n) - u^* \phi(ua_1, a_2, \ldots, a_n) \,\mathrm{d}\,\mu(u).$$

Note that the integrand in the previous line is of the form of line (18) so we may write

$$\phi(a_1, \ldots, a_n)$$

$$= \int_{\mathcal{U}} a_1 u^* \phi(u, a_2, \ldots, a_n) + \sum_{j=1}^{n-1} (-1)^j u^* \phi(u, a_1, \ldots, a_j a_{j+1}, \ldots, a_n)$$

$$+ (-1)^n u^* \phi(u, a_1, \ldots, a_{n-1}) a_n \,\mathrm{d}\,\mu(u)$$

$$= a_1 J_1 \phi(a_2, \ldots, a_n) + \sum_{i=1}^{n-1} (-1)^i J_1 \phi(a_1, \ldots, a_j a_{j+1}, \ldots, a_n)$$

$$+ (-1)^n J_1 \phi(a_1, \ldots, a_{n-1}) a_n$$

$$= \partial J_1 \phi(a_1, \ldots, a_n).$$

Then, extending by linearity, we have that $(\phi - \partial J_1\phi)(a_1, \ldots, a_n) = 0$ whenever $a_1 \in \mathcal{B}$. $\square$

Claim 4.3.4. *Let $\mathcal{B}$ be the C^*-algebra spanned by an amenable group $\mathcal{U}$ of unitaries in a unital C^*-subalgebra $\mathcal{A}$, and let $\mathcal{W}$ be a dual Banach $\mathcal{A}$-bimodule. Additionally, suppose that J_k has been constructed such that*

$$(\phi - \partial J_k\phi)(a_1, \ldots, a_n) = 0 \textit{ if } a_j \in \mathcal{B}, \textit{ for some } j \in \{1, \ldots, k\}\,.$$

Then, J_{k+1} is continuous and $(\phi - \partial J_{k+1}\phi)(a_1, \ldots, a_n) = 0$ if $a_j \in \mathcal{B}$, for some $j \in \{1, \ldots, k\}$.

Proof. First, note that for all $1 \leq k \leq n$,

$$\|G_k\phi(a_1, \ldots, a_{n-1})\| = \left\| \int_{\mathcal{U}} \phi(a_1, \ldots, a_k u^*, u, a_{k+1}, \ldots, a_{n-1}) \,\mathrm{d}\,\mu(u) \right\|$$

$$\leq \sup_{u \in \mathcal{U}} \|\phi(a_1, \ldots, a_k u^*, u, a_{k+1}, \ldots, a_{n-1})\| \leq \|\phi\|\,\|a_1\| \cdots \|a_{n-1}\|$$

so that, G_k is bounded with $\|G_k\| \leq 1$ and so is continuous. Thus, by the construction of J_{k+1}, our map is continuous.

Recall that $J_{k+1} = J_k + (-1)^k G_k(I - \partial J_k)$. By inductive hypothesis,

$$G_k(\phi - \partial J_k\phi)(a_1, \ldots, a_{n-1}) = 0 \text{ if any one of } a_1, \ldots, a_k \in \mathcal{B}$$

since we are averaging over a zero map. Thus, if $a_i \in \mathcal{B}$, for some $i \in \{2, \ldots, k\}$, then

$$\partial G_k(\phi - \partial J_k\phi)(a_1, \ldots, a_n)$$

$$= a_1 G_k(\phi - \partial J_k \phi)\Big((a_2)_1, \ldots, (a_i)_{i-1}, \ldots, (a_n)_{n-1}\Big)$$

$$+ \sum_{j=1}^{i-2}(-1)^j G_k(\phi - \partial J_k \phi)\Big((a_1)_1, \ldots, (a_j a_{j+1})_j, \ldots, (a_i)_{i-1}, \ldots, (a_n)_{n-1}\Big)$$

$$+ (-1)^{i-1} G_k(\phi - \partial J_k \phi)\Big((a_1)_1, \ldots, (a_{i-1} a_i)_{i-1}, \ldots, (a_n)_{n-1}\Big) \tag{19}$$

$$+ (-1)^{i} G_k(\phi - \partial J_k \phi)\Big((a_1)_1, \ldots, (a_i a_{i+1})_i, \ldots, (a_n)_{n-1}\Big) \tag{20}$$

$$+ \sum_{j=i+1}^{n}(-1)^j G_k(\phi - \partial J_k \phi)\Big((a_1)_1, \ldots, (a_i)_i, \ldots, (a_j a_{j+1})_j, \ldots, (a_n)_{n-1}\Big)$$

$$+(-1)^{n+1} G_k(\phi - \partial J_k \phi)(a_1, \ldots, a_i, \ldots, a_{n-1})a_n.$$

Note that every term in the above sum is zero except for (19) and (20) since $a_i \in \mathcal{B}$ is in the ith or $(i-1)$th coordinate. Hence,

$$\partial G_k(\phi - \partial J_k \phi)(a_1, \ldots, a_n)$$

$$= (-1)^{i-1} G_k(\phi - \partial J_k \phi)\Big((a_1)_1, \ldots, (a_{i-1} a_i)_{i-1}, \ldots, (a_n)_{n-1}\Big)$$

$$+(-1)^{i} G_k(\phi - \partial J_k \phi)\Big((a_1)_1, \ldots, (a_i a_{i+1})_i, \ldots, (a_n)_{n-1}\Big).$$

Next, recall that ϕ is a cocycle, so that

$$\partial(\phi - \partial J_k \phi) = \partial\phi - \partial^2 J_k \phi = 0$$

since ∂^2 is a zero map. Moreover, by inductive hypothesis $(\phi - \partial J_k \phi)(a_1, \ldots, a_n) = 0$ since $a_i \in \mathcal{U}$, and $i \leq k$. Thus, by Lemma 4.3.1, for all $u \in \mathcal{U}$ we have

$$(\phi - \partial J_k \phi)\Big((a_1)_1, \ldots, (a_{i-1} a_i)_{i-1}, \ldots, a_k u^*, u, \ldots, (a_n)_n\Big)$$

$$= (\phi - \partial J_k \phi)\Big((a_1)_1, \ldots, (a_i a_{i+1})_i, \ldots, a_k u^*, u, \ldots, (a_n)_n\Big)$$

and so, using the linearity of the averaging operator, we have that

$$\partial G_k(\phi - \partial J_k\phi)(a_1, \ldots, a_n) = 0$$

whenever $a_i \in \mathcal{B}$, for some $i \in \{2, \ldots, k\}$. Furthermore, a similar calculation shows that if $a_1 \in \mathcal{B}$ we have that

$$\partial G_k(\phi - \partial J_k\phi)(a_1, \ldots, a_n) = 0.$$

Then, since

$$(\phi - \partial J_{k+1}\phi)(a_1, \ldots, a_n) = \Big(\phi - \partial\big(J_k\phi + (-1)^k G_k(\phi - \partial J_k\phi)\big)\Big)(a_1, \ldots, a_n)$$

$$= (\phi - \partial J_k\phi)(a_1, \ldots, a_n) - (-1)^k \partial G_k(\phi - \partial J_k\phi)(a_1, \ldots, a_n), \tag{21}$$

we have that

$$(\phi - \partial J_{k+1}\phi)(a_1, \ldots, a_n) = 0$$

whenever $a_i \in \mathcal{B}$, for some $i \in \{1, \ldots, k\}$. □

Claim 4.3.5. *Let $\mathcal{B}$ be the C^*-algebra spanned by an amenable group $\mathcal{U}$ of unitaries in a unital C^*-subalgebra $\mathcal{A}$, and let $\mathcal{W}$ be a dual Banach $\mathcal{A}$-bimodule. Additionally, suppose that J_k has been constructed such that $(\phi - \partial J_k\phi)(a_1, \ldots, a_n) = 0$ if $a_j \in \mathcal{B}$, for some $j \in \{1, \ldots, k\}$ and that*

$$\|J_k\| \leq \frac{(n+2)^k - 1}{n+1}.$$

Then, $(\phi - \partial J_{k+1}\phi)(a_1, \ldots, a_n) = 0$ if $a_{k+1} \in \mathcal{B}$. Moreover,

$$\|J_{k+1}\| \leq 1 + (n+2)\|J_k\| \leq \frac{(n+2)^{k+1} - 1}{n+1}.$$

Proof. First, note that for all $u \in \mathcal{U}$ we have

$$0 = \partial(\phi - \partial J_k\phi)\Big(a_1, \ldots, (a_k u^*)_k, (u)_{k+1}, (a_{k+1})_{k+2}, \ldots, (a_n)_{n+1}\Big)$$

$$= a_1(\phi - \partial J_k\phi)\Big((a_2)_1, \ldots, (a_k u^*)_{k-1}, (u)_k, \ldots, a_n\Big) \tag{22}$$

$$+\sum_{j=1}^{k-1}(-1)^j(\phi - \partial J_k\phi)\Big(a_1, \ldots, (a_j a_{j+1})_j, \ldots, (a_k u^*)_{k-1}, (u)_k, \ldots, a_n\Big) \tag{23}$$

$$+(-1)^k(\phi - \partial J_k\phi)(a_1, \ldots, a_k, \ldots, a_n)$$

$$+(-1)^{k+1}(\phi - \partial J_k\phi)(a_1, \ldots, a_k u^*, u a_{k+1}, \ldots, a_n)$$

$$+\sum_{j=k+1}^{n-1}(-1)^{j+1}(\phi - \partial J_k\phi)\Big((a_1)_1, \ldots, (a_k u^*)_k, (u)_{k+1}, \ldots, (a_j a_{j+1})_{j+1}, \ldots, a_n\Big)$$

$$+(-1)^{n+1}(\phi - \partial J_k\phi)\Big((a_1)_1, \ldots, (a_k u^*)_k, (u)_{k+1}, \ldots, (a_{n-1})_n\Big)a_n.$$

By inductive hypothesis lines (22) and (23) are zero. Thus, rearranging we have

$$(-1)^k(\phi - \partial J_k\phi)(a_1, \ldots, a_k, \ldots, a_n)$$

$$+(-1)^{k+1}(\phi - \partial J_k\phi)(a_1, \ldots, a_k u^*, u a_{k+1}, \ldots, a_n)$$

$$= \sum_{j=k+1}^{n-1}(-1)^j(\phi - \partial J_k\phi)\Big((a_1)_1, \ldots, (a_k u^*)_k, (u)_{k+1}, \ldots, (a_j a_{j+1})_{j+1}, \ldots, (a_n)_n\Big)$$

$$+ (-1)^n(\phi - \partial J_k\phi)\Big((a_1)_1, \ldots, (a_k u^*)_k, (u)_{k+1}, \ldots, (a_{n-1})_n\Big)a_n. \tag{24}$$

Next, to show that

$$(\phi - \partial J_{k+1}\phi)(a_1, \ldots, a_n) = 0 \text{ when } a_{k+1} \in \mathcal{B}$$

it suffices to show when $a_{k+1} \in \mathcal{U}$ then extending by linearity. Observe that,

$$\partial G_k(\phi - \partial J_k\phi)(a_1, \ldots, a_n)$$

$$= \sum_{j=k}^{n-1} (-1)^j G_k(\phi - \partial J_k \phi)(a_1, \ldots, a_j a_{j+1}, \ldots, a_n)$$

$$+(-1)^n G_k(\phi - \partial J_k \phi)(a_1, \ldots, a_{n-1}) a_n$$

since the boundary operator shifts $a_{k+1} \in \mathcal{U}$ into the k'th coordinate for $j < k$. Then, using the linearity of our averaging operator, we may write.

$$\partial G_k(\phi - \partial J_k \phi)(a_1, \ldots, a_n)$$

$$= \int_{\mathcal{U}} (-1)^k (\phi - \partial J_k \phi)\Big(a_1, \ldots, (a_k a_{k+1} u^*)_k, (u)_{k+1}, a_{k+2}, \ldots, a_n\Big)$$

$$\sum_{j=k+1}^{n-1} (-1)^j (\phi - \partial J_k \phi)\Big(a_1, \ldots, (a_k u^*)_k, (u)_{k+1}, (a_{k+1})_{k+2}, \ldots, (a_j a_{j+1})_{j+1}, \ldots, a_n\Big)$$

$$+(-1)^n (\phi - \partial J_k \phi)(a_1, \ldots, a_k u^*, u, \ldots, a_{n-1}) a_n \ \mathrm{d}\, \mu(u).$$

Then, using equation (24), we have

$$\partial G_k(\phi - \partial J_k \phi)(a_1, \ldots, a_n)$$

$$= \int_{\mathcal{U}} (-1)^k (\phi - \partial J_k \phi)\Big(a_1, \ldots, (a_k a_{k+1} u^*)_k, (u)_{k+1}, a_{k+2}, \ldots, a_n\Big) \ \mathrm{d}\, \mu(u) \tag{25}$$

$$+(-1)^k (\phi - \partial J_k \phi)(a_1, \ldots, a_k, \ldots, a_n)$$

$$+ \int_{\mathcal{U}} (-1)^{k+1} (\phi - \partial J_k \phi)\Big(a_1, \ldots, (a_k u^*)_k, (u a_{k+1})_{k+1}, \ldots, a_n\Big) \ \mathrm{d}\, \mu(u).$$

Hence, making the "change of variables" $v^* = a_{k+1} u^*$ in line (25) we see that

$$\partial G_k(\phi - \partial J_k \phi)(a_1, \ldots, a_n) = (-1)^k (\phi - \partial J_k \phi)(a_1, \ldots, a_k, \ldots, a_n)$$

and so using equation (21)

$$(\phi - \partial J_{k+1} \phi)(a_1, \ldots, a_n) = 0 \ \text{ whenever } \ a_{k+1} \in \mathcal{B}.$$

Lastly, using (21) again,

$$\|J_{k+1}\| \le 1 + (n+2)\,\|J_k\| \le \frac{(n+2)^{k+1}-1}{n+1}. \quad \square$$

Remark 4.3.6. If our averaging operator, i.e. the "integral" over the unitary group $\mathcal{U}$, converges in the weak* topology of the dual normal $\mathcal{A}$-bimodule $\mathcal{W}$ to an element in the subdual $\mathcal{V}$ of $\mathcal{W}$ for all $\phi \in \mathscr{L}^n(\mathcal{A}, \mathcal{V})$ then we may replace $\mathcal{W}$ with $\mathcal{V}$ everywhere in the above proof.

Lemma 4.3.7 ([19] Lemma 3.2.6)**.** *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{V}$ be a dual $\mathcal{A}$-bimodule. Suppose that $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$ generated by an amenable group $\mathcal{U}$ of unitaries. Then there is a continuous surjective linear projection $Q_n : \mathscr{L}_c^n(\mathcal{A}, \mathcal{V}) \to \mathscr{L}_c^n(\mathcal{A}, \mathcal{V} : \mathcal{B})$ such that $\partial Q_{n-1} = Q_n \partial$ and $\|Q_n\| = 1$.*

Proof. First we define maps $R_j : \mathscr{L}_c^n(\mathcal{A}, \mathcal{V}) \to \mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$ by

(i) $(R_0\phi)(a_1, \ldots, a_n) = \int_{\mathcal{U}} u^*\phi(ua_1, a_2, \ldots, a_n)\,\mathrm{d}\,\mu(u)$,

(ii) $(R_k\phi)(a_1, \ldots, a_n) = \int_{\mathcal{U}} \phi(a_1, \ldots, a_k u^*, ua_{k+1}, \ldots, a_n)\,\mathrm{d}\,\mu(u)$ where $1 \le k < n$, and

(iii) $(R_n\phi)(a_1, \ldots, a_n) = \int_{\mathcal{U}} \phi(a_1, \ldots, a_n u^*)u\,\mathrm{d}\,\mu(u)$.

Since the averaging operator is contractive we see that each R_j is continuous with norm less than 1. Next, using the invariance of the averaging operator, for $v \in \mathcal{U}$ we have that

$$\begin{gathered}(R_k\phi)(a_1, \ldots, a_k v, a_{k+1}, \ldots, a_n) \\ = \int_{\mathcal{U}} \phi(a_1, \ldots, a_k v u^*, ua_{k+1}, a_n)\,\mathrm{d}\,\mu(u) = \int_{\mathcal{U}} \phi(a_1, \ldots, a_k w^*, wva_{k+1}, a_n)\,\mathrm{d}\,\mu(w) \\ = (R_k\phi)(a_1, \ldots, a_k, va_{k+1}, \ldots, a_n)\end{gathered}$$

where we have made the 'change of variables' $vu^* = w^*$. Then, extending by linearity, we see that $R_k\phi$ has the $\mathcal{B}$-modularity property across the kth and

$k+1$ positions. Likewise, a similar calculation shows that $R_0\phi$ and $R_n\phi$ have the $\mathcal{B}$-modularity property across the 0th and the first positions and the nth and the $n+1$ positions respectively. Next, define

$$Q_n = R_n \circ R_{n-1} \circ \cdots \circ R_0.$$

Note that an elementary induction shows that $Q_n\phi$ is $\mathcal{B}$-multimodular.

To show that Q_n is a projection suppose that $\phi \in \mathscr{L}_c^n(\mathcal{A}, \mathcal{V} : \mathcal{B})$. Then,

$$\begin{aligned} &R_k\phi(a_1, \ldots, a_n) \\ &= \int_{\mathcal{U}} \phi(a_1, \ldots, a_k u^*, u a_{k+1}, \ldots, a_n) \,\mathrm{d}\,\mu(u) \\ &= \phi(a_1, \ldots, a_n). \end{aligned}$$

by the multimodularity of ϕ and since the averaging operator acts as the identity on constant functions. Thus, since each R_k acts as the identity on multimodular maps, Q_n acts as the identity on multimodular maps also.

Lastly we show that $Q_n\partial = \partial Q_{n-1}$ by inducting on k for $(R_k \circ \cdots \circ R_0)\partial\phi$. Observe that

$$\begin{aligned} R_0\partial\phi(a_1, \ldots, a_{n+1}) &= \int_{\mathcal{U}_0} u_0^*(\partial\phi)(u_0 a_1, \ldots, a_{n+1}) \,\mathrm{d}\,\mu(u_0) \\ &= a_1\phi(a_2, \ldots, a_{n+1}) \\ &+ \sum_{j=1}^{n}(-1)^j \int_{\mathcal{U}_0} u_0^*\phi(u_0 a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) \,\mathrm{d}\,\mu(u_0) \\ &+ (-1)^{n+1} \int_{\mathcal{U}_0} u_0^*\phi(u_0 a_1, \ldots, a_n) \,\mathrm{d}\,\mu(u_0) a_{n+1}. \end{aligned}$$

Next, letting R_1 act on both sides we have

$$R_1 \cdot R_0\partial\phi(a_1, \ldots, a_{n+1}) = \int_{\mathcal{U}_1} (R_0\partial\phi)(a_1 u_1^*, u_1 a_2, \ldots, a_{n+1}) \,\mathrm{d}\,\mu(u_1)$$

$$= a_1 \int_{\mathcal{U}_1} u_1^* \phi(u_1 a_2, \ldots, a_{n+1}) \,\mathrm{d}\,\mu(u_1) - \int_{\mathcal{U}_0} u_0^* \phi(u_0 a_1 a_2, \ldots, a_{n+1} \,\mathrm{d}\,\mu(u_0)$$

$$+ \sum_{j=2}^{n} (-1)^j \int_{\mathcal{U}_1} \int_{\mathcal{U}_0} u_0^* \phi(u_0 a_1 u_1^*, u_1 a_2, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) \,\mathrm{d}\,\mu(u_0) \,\mathrm{d}\,\mu(u_1)$$

$$+ (-1)^{n+1} \int_{\mathcal{U}_1} \int_{\mathcal{U}_0} u_0^* \phi(u_0 a_1 u_1^*, u_1 a_2, \ldots, a_n) \,\mathrm{d}\,\mu(u_0) \,\mathrm{d}\,\mu(u_1) a_{n+1}$$

so that

$$(R_1 \circ R_0) \partial \phi(a_1, \ldots, a_{n+1})$$

$$= a_1 (R_0 \phi)(a_2, \ldots, a_{n+1}) - (R_0 \phi)(a_1 a_2, \ldots, a_{n+1})$$

$$+ \sum_{j=2}^{n} (-1)^j (R_1 \circ R_0) \phi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1})$$

$$+ (-1)^{n+1} (R_1 \circ R_0) \phi(a_1, \ldots, a_n) a_{n+1}$$

Before we proceed to the inductive step note that,

$$R_j \cdot (R_j \phi)(a_1, \ldots, a_n) = \int_{\mathcal{U}_1} (R_j \phi)(a_1, \ldots, a_j u_1^*, u_1 a_{j+1}, \ldots, a_n) \,\mathrm{d}\,\mu(u_1)$$

$$= \int_{\mathcal{U}_1} \int_{\mathcal{U}_2} \phi(a_1, \ldots, a_j u_1^* u_2^*, u_2 u_1 a_{j+1}, \ldots, a_n) \,\mathrm{d}\,\mu(u_1) \,\mathrm{d}\,\mu(u_2).$$

So by the invariance of the averaging operator we have

$$R_j \cdot (R_j \phi)(a_1, \ldots, a_n) = R_j \phi(a_1, \ldots, a_n). \tag{26}$$

Now suppose that we have carried out this process to the kth step. That is,

$$(R_k \circ \cdots \circ R_0) \partial \phi(a_1, \ldots, a_{n+1})$$

$$= a_1(R_{k-1} \circ \cdots \circ R_0)\phi(a_2, \ldots, a_{n+1})$$

$$+ \sum_{j=1}^{k-1} (-1)^j (R_{k-1} \circ \cdots \circ R_0)\phi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1})$$

$$+ \sum_{j=k}^{n} (-1)^j (R_k \circ \cdots \circ R_0)\phi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1})$$

$$+(-1)^{n+1}(R_k \circ \cdots \circ R_0)\phi(a_1, \ldots, a_n)a_{n+1}.$$

Then, if we let R_{k+1} act on both sides, we have

$$R_{k+1} \cdot (R_k \circ \cdots \circ R_0)\partial\phi(a_1, \ldots, a_{n+1})$$

$$= \int_{\mathcal{U}} (R_k \circ \cdots \circ R_0)\partial\phi(a_1, \ldots, a_{k+1}u^*, ua_{k+2}, \ldots, a_{n+1}) \, \mathrm{d}\,\mu(u)$$

$$= a_1 \int_{\mathcal{U}} (R_{k-1} \circ \cdots \circ R_0)\phi\Big(a_2, \ldots, (a_{k+1}u^*)_k, (ua_{k+2})_{k+1}, \ldots, a_{n+1}\Big) \, \mathrm{d}\,\mu(u)$$

$$+ \sum_{j=1}^{k-1} (-1)^j \int_{\mathcal{U}} (R_{k-1} \circ \cdots \circ R_0)\phi\Big(a_1, \ldots, a_j a_{j+1}, \ldots, (a_{k+1}u^*)_k, (ua_{k+2})_{k+1}, \ldots, a_{n+1}\Big) \, \mathrm{d}\,\mu(u)$$

$$+(-1)^k \int_{\mathcal{U}} (R_k \circ \cdots \circ R_0)\phi\Big(a_1, \ldots, (a_k a_{k+1}u^*)_k, (ua_{k+2})_{k+1}, \ldots, a_{n+1}\Big) \, \mathrm{d}\,\mu(u)$$

$$+ \sum_{j=k+1}^{n} (-1)^j \int_{\mathcal{U}} (R_k \circ \cdots \circ R_0)\phi\Big(a_1, \ldots, a_{k+1}u^*, ua_{k+2}, \ldots, a_j a_{j+1}, \ldots, a_{n+1}\Big) \, \mathrm{d}\,\mu(u)$$

$$+(-1)^{n+1} \int_{\mathcal{U}} (R_k \circ \cdots \circ R_0)\phi\Big(a_1, \ldots, a_{k+1}u^*, ua_{k+2}, \ldots, a_n\Big) \, \mathrm{d}\,\mu(u)a_{n+1}.$$

Then using line (26) we arrive at

$$(R_{k+1} \circ \cdots \circ R_0)\partial\phi(a_1, \ldots, a_{n+1})$$

$$
\begin{aligned}
&= a_1(R_k \circ \cdots \circ R_0)\phi(a_2, \ldots, a_{n+1}) \\
&+ \sum_{j=1}^{k} (-1)^j (R_k \circ \cdots \circ R_0)\phi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) \\
&+ \sum_{j=k+1}^{n} (-1)^j (R_{k+1} \circ \cdots \circ R_0)\phi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) \\
&+ (-1)^{n+1}(R_{k+1} \circ \cdots \circ R_0)\phi(a_1, \ldots, a_n)a_{n+1}.
\end{aligned}
$$

Finally, by setting $k = n-1$ and applying R_n once more, using line (26) once more we arrive at $Q_n\partial = \partial Q_{n-1}$. □

We conclude this section with a theorem that will be useful in the next section.

Theorem 4.3.8 ([19] Theorem 3.2.7)**.** *Let $\mathcal{B}$ be the C^*-algebra generated by an amenable group $\mathcal{U}$ of unitaries in a unital C^*-algebra $\mathcal{A}$, and let $\mathcal{V}$ be a dual $\mathcal{A}$-bimodule. Then*

$$H_c^n(\mathcal{A}, \mathcal{V}) \cong H_c^n(\mathcal{A}, \mathcal{V} : \mathcal{B})$$

for all $n \in \mathbb{N}$ with isomorphism induced by the natural embedding

$$\mathscr{L}_c^n(\mathcal{A}, \mathcal{V} : \mathcal{B}) \hookrightarrow \mathscr{L}_c^n(\mathcal{A}, \mathcal{V}).$$

Proof. Clearly, the natural embedding $\mathscr{L}_c^n(\mathcal{A}, \mathcal{V} : \mathcal{B}) \hookrightarrow \mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$ induces a homomorphism $H_c^n(\mathcal{A}, \mathcal{V} : \mathcal{B}) \to H_c^n(\mathcal{A}, \mathcal{V})$. By Lemma 4.3.2 this map is surjective. Furthermore, if $\phi \in \mathscr{L}_c^n(\mathcal{A}, \mathcal{V} : \mathcal{B})$ and $\psi \in \mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$ is such that $\phi = \partial\psi$, then with Q_n as in Lemma 4.3.7,

$$\phi = Q_n\phi = Q_n\partial\psi = \partial Q_{n-1}\psi \;\text{ where }\; Q_{n-1}\psi \in \mathscr{L}_c^{n-1}(\mathcal{A}, \mathcal{V} : \mathcal{B})$$

and so our map is injective. □

5 A Relation Between Cohomologies

The goal of this section is to prove the following theorem.

Theorem 5.0.1. *If the natural map $H^n_w(C^*_u(X)) \to H^n_c(C^*_u(X))$ is surjective then*

$$H^n_c(C^*_u(X)) = 0.$$

Note that, by [9] Lemma 3, all bounded derivations are weakly continuous. Thus, the natural map $H^1_w(C^*_u(X)) \to H^1_c(C^*_u(X))$ is automatically surjective.

5.1 A Corollary of Braga and Farah's Lemma, Multilinear Version

Definition 5.1.1 (separately symmetrically summable)**.** For a finite sequence of countable index sets $\{I_n\}_{n=1}^N$, $N < \infty$, a uniformly bounded family of operators $(a_{(i_1,\ldots,i_N)})_{(\bar{i}\in\prod_{n=1}^N I_n)} \subseteq C^*_u(X)$ is N *separately symmetrically summable* if the following condition holds.

For any $(1 \leq k \leq N)$, and for each fixed

$$\left\{\lambda^{(1)},\ldots,\lambda^{(k-1)},\lambda^{(k+1)},\ldots,\lambda^{(N)}\right\} \in \prod_{\substack{n=1\\ n\neq k}}^{N} \mathbb{D}^{I_n}$$

the sum

$$\sum_{i_k\in I_k} \lambda^{(k)}_{i_k} a_{(\lambda^{(1)},\ldots,\lambda^{(k-1)},i_k,\lambda^{(k+1)},\ldots,\lambda^{(N)})}$$

converges in the weak operator topology to an element

$$a_{(\lambda^{(1)},\ldots,\lambda^{(k)},\ldots,\lambda^{(N)})} \in C^*_u(X).$$

Additionally,

$$\text{for all } \left\{\lambda^{(1)}, \ldots, \lambda^{(N)}\right\} \in \prod_{n=1}^{N} \mathbb{D}^{I_n}, \quad a_{(\lambda^{(1)},\ldots,\lambda^{(N)})} \subseteq C_u^*(X)$$

is a uniformly bounded family of operators.

Moreover, if $(a_{(i_1,\ldots,i_{N+1})})_{(\bar{i}\in\prod_{n=1}^{N+1} I_n)}$ is $(N+1)$ separately symmetrically summable, then for any fixed $\eta \in \mathbb{D}^{I_{N+1}}$, $(a_{(i_1,\ldots,i_N,\eta)})_{(\bar{i}\in\prod_{n=1}^{N} I_n)}$ is N separately symmetrically summable.

Corollary 5.1.2 (Corollary to Theorem 3.3.2)**.** *Suppose that*

$$(a_{(i_1,\ldots,i_N)})_{(\bar{i}\in\prod_{n=1}^{N} I_n)} \subseteq C_u^*(X)$$

is N separately symmetrically summable. Then for any $\epsilon > 0$ there exists an $r > 0$ such that for all $(\lambda^{(1)}, \ldots, \lambda^{(N)}) \in \prod_{n=1}^{N} \mathbb{D}^{I_n}$, the operator $a_{(\lambda^{(1)},\ldots,\lambda^{(N)})}$ is ϵ-r-approximated.

To prove this corollary we induct on N. However, we will need a definition and a few lemmas first. Note that the base case is handled by Theorem 3.3.2.

Definition 5.1.3. Suppose that $(a_{(i_1,\ldots,i_N)})_{(\bar{i}\in\prod_{n=1}^{N} I_n)} \subseteq C_u^*(X)$ is N separately symmetrically summable. Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N-1)})$. Then for $\eta \in \mathbb{D}^{I_N}$ we let

$$a_{\lambda,\eta} = \sum_{i_N \in I_N} \eta_{i_N} a_{\lambda,i_N}$$

Then for $\epsilon, r > 0$ define

$$U_{\epsilon,r} := \left\{ \eta \in \mathbb{D}^{I_N} \mid a_{\lambda,\eta} \text{ is } \epsilon\text{-}r\text{-approximated for all } \lambda \in \prod_{n=1}^{N-1} \mathbb{D}^{I_n} \right\}.$$

Remark 5.1.4. On the first read it may provide intuition to just consider the $N = 2$ case since the proof of the inductive step is only notationally different.

Suppose that $\epsilon > 0$ is given. If we are considering the $N = 2$ case, and $\{a_{i,j}\}_{i\in I, j\in J}$ is 2 separately symmetrically summable. Then, for each fixed $\eta \in \mathbb{D}^J$, $\{a_{i,\eta}\}_{i\in I}$ symmetrically summable so by Theorem 3.3.2 we may write $\mathbb{D}^J$ as the union in line 27.

For the inductive step, if $(a_{(i_1,\ldots,i_N)})_{(\bar{i}\in\prod_{n=1}^N I_n)}$ is N separately symmetrically summable. Then, for each fixed $\eta \in \mathbb{D}^{I_N}$ we have that $(a_{(i_1,\ldots,i_{N-1},\eta)})_{(\bar{i}\in\prod_{n=1}^{N-1} I_n)}$ is $(N-1)$ separately symmetrically summable. Thus, by inductive hypothesis we may write $\mathbb{D}^{I_N}$ as the union

$$\mathbb{D}^{I_N} = \bigcup_{r=1}^{\infty} U_{\epsilon,r}. \tag{27}$$

As in the original proof we will first show that the sets in Definition 5.1.3 are closed for any N separately symmetrically summable $(a_{(i_1,\ldots,i_N)})_{(\bar{i}\in\prod_{n=1}^N I_n)}$. Then we will show that if $(a_{(i_1,\ldots,i_N)})_{(\bar{i}\in\prod_{n=1}^N I_n)}$ does not satisfy the conclusion of Theorem 5.1.2, there is $\epsilon > 0$ such that for all $r > 0$, $U_{r,\epsilon}$ is nowhere dense in $\mathbb{D}^{I_N}$. As we have the union in line (27), this contradicts the Baire category theorem and we will be done.

Lemma 5.1.5. *Suppose that $(a_{(i_1,\ldots,i_N)})_{(\bar{i}\in\prod_{n=1}^N I_n)}$ is separately symmetrically summable. Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N-1)})$. Then for any $\epsilon, r > 0$ the set $U_{\epsilon,r}$ of Definition 5.1.3 is closed.*

Proof. Assume for contradiction that for some $\epsilon, r > 0$, $U_{\epsilon,r}$ is not closed. Then there exists some $\eta \in \overline{U_{\epsilon,r}} \setminus U_{\epsilon,r}$. As $\eta \notin U_{\epsilon,r}$, there exists a $\lambda \in \prod_{n=1}^{N-1} \mathbb{D}^{I_n}$ such that $a_{\lambda,\eta}$ cannot be ϵ-r-approximated. Fix this λ. With this λ fixed the remainder of the argument proceeds precisely as in Lemma 3.3.6. However, out of an abundance of care, we complete the argument.

Using (the contrapositive of) Lemma 3.3.4, part (i), there exists a finite rank projection $p \in \ell^\infty(X)$ such that $pa_{\lambda,\eta}p$ cannot be ϵ-r-approximated.

Now, for any $\mu \in \mathbb{D}^{I_N}$, the sum $\sum_{i\in I_N} \mu_i a_{\lambda,i}$ defining $a_{\lambda,\mu}$ is weakly convergent. As p is finite rank, this implies that the sum $\sum_{i\in I_N} p\mu_i a_{\lambda,i} p$ is norm convergent (cf. 2.3.2). Hence using Lemma 3.3.5, for any $\delta > 0$ there exists a finite subset F of I_N such that

$$\left\| \sum_{i \in I_N \setminus F} p\mu_i a_{\lambda,i} p \right\| < \delta \tag{28}$$

for all $\mu \in \mathbb{D}^{I_N}$ (and in particular for $\mu = \eta$).

As F is finite, the set

$$\left\{ \mu \in \mathbb{D}^{I_N} \;\middle|\; |F| \max_{i \in F} \|a_{\lambda,i}\| |\mu_i - \eta_i| < \delta \text{ for all } i \in F \right\} \tag{29}$$

is an open neighborhood of η for the product topology. As η is in the closure of $U_{\epsilon,r}$, the set in line (29) thus contains some $\theta \in U_{\epsilon,r}$. Hence in particular $pa_{\lambda,\theta}p$ is ϵ-r-approximated, so there is $b \in \mathbb{C}_u^r[X]$ such that $\|pa_{\lambda,\theta}p - b\| \leq \epsilon$.

Note that

$$\|pa_{\lambda,\eta}p - b\| \leq \|pa_{\lambda,\theta}p - b\| + \|pa_{\lambda,\eta}p - pa_{\lambda,\theta}p\|$$

$$\leq \|pa_{\lambda,\theta}p - b\| + \left\| \sum_{i \in F} (\eta_i - \theta_i) pa_{\lambda,i} p \right\| + \left\| \sum_{i \in I_N \setminus F} \theta_i pa_{\lambda,i} p \right\| + \left\| \sum_{i \in I_N \setminus F} \eta_i pa_{\lambda,i} p \right\|.$$

The first term on the bottom line is bounded above by ϵ by choice of b, the second is bounded above by δ using that θ is in the set in line (29), and the third and fourth terms are bounded above by δ using the estimate in line (28) (which is valid for all elements η of $\mathbb{D}^{I_N}$).

Now, we have shown that for arbitrary $\delta > 0$, we have found $b \in \mathbb{C}_u^r[X]$ such that $\|pa_{\lambda,\eta}p - b\| \leq \epsilon + 3\delta$. Using Lemma 3.3.4, part (ii), this implies that $pa_{\lambda,\eta}p$ can be ϵ-r-approximated. This contradicts our assumption in the first paragraph, so we are done. $\square$

Lemma 5.1.6. *Suppose that* $(a_{(i_1,\ldots,i_N)})_{(\bar{i}\in\prod_{n=1}^{N} I_n)} \subseteq C_u^*(X)$ *is separately symmetrically summable. Let* $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N-1)})$. *Then, for all* $\epsilon > 0$, *for any* $\theta \in \mathbb{D}^{I_N}$, *and any finite* $F \subseteq I_N$ *there exists an* $r > 0$ *such that the sum* $\sum_{i\in F} \theta_i a_{\lambda,i}$ *is* ϵ-r-*approximated.*

Proof. Let F be a finite subset of I_N and $\epsilon > 0$ be given. By supposition, for each i, we may write

$$a_{\lambda,i} = b_{\lambda,i} + c_{\lambda,i} \text{ where } b_{\lambda,i} \in \mathbb{C}_u^{r_i}[X] \text{ and } \|c_{\lambda,i}\| < \tfrac{\epsilon}{|F|}.$$

Let $r = \max_{i\in F}\{r_i\}$ and note that $\sum_{i\in F} \theta_i b_{\lambda,i} \in \mathbb{C}_u^{r}[X]$ for all λ. Additionally,

$$\left\| \sum_{i\in F} \theta_i c_{\lambda,i} \right\| \leq \sum_{i\in F} |\theta_i| \, \|c_{\lambda,i}\| < \epsilon.$$

Hence, $\sum_{i\in F} \theta_i a_{\lambda,i}$ is ϵ-r-approximated for all $\lambda \in \prod_{n=1}^{N-1} \mathbb{D}^{I_n}$. □

Lemma 5.1.7. *Suppose that* $(a_{(i_1,\ldots,i_N)})_{(\bar{i}\in\prod_{n=1}^{N} I_n)}$ *is a separately symmetrically summable collection of operators in* $C_u^*(X)$ *that does not satisfy the conclusion of Lemma 5.1.2. Additionally, let* $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N-1)})$. *Then there is an* $\epsilon > 0$ *so that for all* $r > 0$ *and all finite subsets* $F \subseteq I_N$ *there exists* $\eta \in \mathbb{D}^{I_N}$ *such that* $\sum_{i\in I_N\setminus F} \eta_i a_{\lambda,i}$ *cannot be* ϵ-r-*approximated.*

Proof. Let $(a_{(i_1,\ldots,i_N)})_{(\bar{i}\in\prod_{n=1}^{N} I_n)}$ be as in the statement. Then there exists $\delta > 0$ such that for all $r > 0$ there exists $(\lambda, \eta) \in \prod_{n=1}^{N-1} \mathbb{D}^{I_n} \times \mathbb{D}^{I_N}$ such that $a_{\lambda,\eta}$ is not δ-r-approximable. Fix this λ. Assume for contradiction that the conclusion of the lemma fails. Then there exists $s > 0$ and a finite subset F of I_N such that for all $\xi \in \mathbb{D}^{I_N}$ we have that $\sum_{i\in I_N\setminus F} \xi_i a_{\lambda,i}$ is $\delta/2$-s-approximated. As F is finite, by Lemma 5.1.6 there is a $t > 0$ such that every element of

$$\left\{ \sum_{i\in F} \xi_i a_{\lambda,i} \;\middle|\; \xi \in \mathbb{D}^{I_N} \right\}$$

can be $\delta/2$-t-approximated. Now, for arbitrary $\xi \in \mathbb{D}^{I_N}$,

$$a_{\lambda,\xi} = \sum_{i \in F} \xi_i a_{\lambda,i} + \sum_{i \in I_N \setminus F} \xi_i a_{\lambda,i};$$

as the first term above can be $\delta/2$-s-approximated, and as the second can be $\delta/2$-t-approximated, this implies that $a_{\lambda,\xi}$ can be δ-$\max\{s,t\}$-approximated. As ξ was arbitrary, this contradicts the first sentence in the proof, and we are done. □

As stated at the end of Remark 5.1.4, the following lemma completes the proof of Corollary 5.1.2.

Lemma 5.1.8. *Suppose that $(a_{(i_1,\dots,i_N)})_{(\bar{i} \in \prod_{n=1}^N I_n)}$ is a separately symmetrically summable collection of operators in $C_u^*(X)$ that does not satisfy the conclusion of Lemma 5.1.2. Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(N-1)})$. Then there is $\epsilon > 0$ such that for each $r > 0$ the set $U_{\epsilon,r}$ of Definition 5.1.3 is nowhere dense in $\mathbb{D}^{I_N}$.*

Proof. Let $(a_{(i_1,\dots,i_N)})_{(\bar{i} \in \prod_{n=1}^N I_n)}$ be as in the statement. Then there exists $\delta > 0$ such that for all $r > 0$ there exists $(\lambda, \eta) \in \left(\prod_{n=1}^{N-1} \mathbb{D}^{I_n}\right) \times \mathbb{D}^{I_N}$ such that $a_{\lambda,\eta}$ is not δ-r-approximable. Fix this λ. Let $\epsilon' > 0$ have the property from Lemma 5.1.7. We claim that $\epsilon := \epsilon'/2$ has the property required for this lemma. Assume for contradiction that for some $r > 0$, $U_{\epsilon,r}$ is not nowhere dense. Lemma 5.1.5 implies that $U_{\epsilon,r}$ is closed, and so it contains a point ξ in its interior. Then by definition of the product topology there exists a finite set $F \subseteq I_N$ and $\delta > 0$ such that the set

$$V := \{\nu \in \mathbb{D}^{I_N} \mid |\xi_i - \nu_i| < \delta \text{ for all } i \in F\} \text{ is contained in } U_{\epsilon,r}. \tag{30}$$

Note that the element $\sum_{i \in F} \xi_i a_{\lambda,i}$ is in $C_u^*(X)$ by assumption, so can be ϵ-s-approximated for some s; let $b_{\xi,\lambda} \in \mathbb{C}_u^s[X]$ be such that $\|\sum_{i \in F} \xi_i a_{\lambda,i} - b_{\lambda,\xi}\| \leq \epsilon$. On the other hand, Lemma 5.1.7 gives us $\mu \in \mathbb{D}^{I_N}$ so that

$\sum_{i \in I \setminus F} \mu_i a_{\lambda,i}$ cannot be ϵ'-$\max\{r,s\}$-approximated. We may further assume that $\mu_i = 0$ for $i \in F$. Define $\theta \in \mathbb{D}^I$ by

$$\theta_i := \begin{cases} \xi_i & i \in F \\ \mu_i & i \notin F \end{cases}$$

Then θ is clearly in the set V of line (30), and so $a_{\lambda,\theta}$ is ϵ-r-approximated. Let then $b_{\lambda,\theta} \in \mathbb{C}_u^r[X]$ be such that $\|a_{\lambda,\theta} - b_{\lambda,\theta}\| \leq \epsilon$. We then see that

$$\|a_{\lambda,\mu} - (b_{\lambda,\theta} - b_{\lambda,\xi})\| \leq \|a_{\lambda,\mu} - a_{\lambda,\theta} + b_{\lambda,\xi}\| + \|a_{\lambda,\theta} - b_{\lambda,\theta}\|$$

$$\leq \left\| b_{\lambda,\xi} - \sum_{i \in F} \xi_i a_{\lambda,i} \right\| + \|a_{\lambda,\theta} - b_{\lambda,\theta}\|$$

The terms on the bottom row are each less than ϵ by choice of $b_{\lambda,\xi}$ and $b_{\lambda,\theta}$, and so $\|a_{\lambda,\mu} - (b_{\lambda,\theta} - b_{\lambda,\xi})\| \leq 2\epsilon = \epsilon'$. As $b_{\lambda,\xi} + b_{\lambda,\theta}$ has propagation at most $\max\{r,s\}$, this contradicts the assumption that $a_{\lambda,\mu}$ cannot be ϵ'-$\max\{r,s\}$-approximated, so we are done. □

5.2 Proof of Theorem 5.0.1

For notational convenience throughout we let: $A = C_u^*(X)$, $\mathscr{B} = \mathscr{B}(\ell^2(X))$, and $\ell = \ell^\infty(X)$. Recall that Theorem 5.0.1 states that if there is a surjection $H_w^n(A) \to H_c^n(A)$ then $H_c^n(A : \ell) \cong H_c^n(A)$. As a first step towards this goal we show that $H_c^n(A, \mathscr{B} : \ell) \cong H_c^n(A : \ell)$ in the following lemma.

Lemma 5.2.1. *Let $\phi \in \mathscr{L}^n(A, \mathscr{B} : \ell)$. Then ϕ takes image in the uniform Roe algebra; that is, $\mathscr{L}^n(A, \mathscr{B} : \ell) = \mathscr{L}^n(A : \ell)$.*

Proof. Let $\phi \in \mathscr{L}^n(A, \mathscr{B} : \ell)$, $(x_1, \ldots, x_n) \in A^n$, and $0 < \epsilon \leq 1$ be given. Set $M = \max\{\|x_i\|\} + 1$ and note that since each $x_i \in A$ we may write each x_i as

$$x_i = a_i + b_i \text{ where } a_i \in \mathbb{C}_u^{r_i}[X] \text{ and } \|b_i\| < \min\left\{ \frac{\epsilon}{n\,\|\phi\|\,M^n},\ \epsilon \right\}.$$

Moreover, we have that $\|a_i\| < M$. Next, since ϕ is multilinear we may write

$$\phi(x_1, \ldots, x_n) = \phi(a_1, \ldots, a_n) + \phi(a_1, \ldots, a_{n-1}, b_n) + \phi(a_1, \ldots, a_{n-2}, b_{n-1}, x_n) +$$

$$\cdots + \phi(a_1, b_2, x_3, \ldots, x_n) + \phi(b_1, x_2, \ldots, x_n)$$

Observe that every term but the first in this expansion has a b_i in a single coordinate and either a_i's or x_i's in the remaining coordinates. Thus, the norm for each of the terms with a b_k in the kth coordinate is bounded by

$$\|\phi\| \left(\prod_{i=1}^{n} M \right) \|b_k\| < \frac{\epsilon}{n}$$

Hence, it is enough to show that $\phi(a_1, \ldots, a_n) \in \mathbb{C}_u^{n \cdot r}[X]$ where $r = \max\{r_i\}$.

To show this let p_x be the projection onto the span of the Dirac mass at x, and let $B_x(r)$ denote the closed ball of radius r centered at x. We then define

$$p_{B_x(r)} := \sum_{k \in B_x(r)} p_k.$$

Note that, the sum defining $p_{B_x(r)}$ is finite for any given $r \in \mathbb{N}$ since X has bounded geometry. Next, for any fixed $x \in X$,

$$p_x a_1 = p_x a_1 p_{B_x(r)} \quad \text{and} \quad p_{B_x((i-1)\cdot r)} a_i = p_{B_x((i-1)\cdot r)} a_i p_{B_x(i \cdot r)} \tag{31}$$

since each a_i has propagation less than r. Next, fix $x, y \in X$ such that $d(x, y) > n \cdot r$ and observe that

$$p_x \phi(a_1, \ldots, a_n) p_y = \phi(p_x a_1, \ldots, a_n p_y)$$

$$= \phi(p_x a_1 p_{B_x(r)}, \ldots, a_n p_y) = \phi(p_x a_1 p_{B_x(r)}, p_{B_x(r)} a_2, \ldots, a_n p_y)$$

where on the left hand we have used line (31) and on the right hand side we use that ϕ is $\ell^\infty(X)$-multimodular.

Continuing this process $n-1$ times we arrive at

$$p_x\phi(a_1,\ldots,a_n)p_y$$

$$= \phi(p_x a_1 p_{B_x(r)},\ldots,p_{B_x((i-1)\cdot r)}a_i p_{B_x(i\cdot r)}\ldots,p_{B_x((n-1)\cdot r)}a_n p_y).$$

Observe that for any $k \in B_x((n-1)\cdot r)$,

$$d(k,y) \geq d(x,y) - d(x,k) \geq d(x,y) - (n-1)\cdot r > n\cdot r - (n-1)\cdot r = r,$$

and so

$$p_{B_x((n-1)\cdot r)}a_n p_y = 0 \text{ since } a_n \in \mathbb{C}_u^r[X].$$

Thus,

$$p_x\phi(a_1,\ldots,a_n)p_y = 0$$

and since $x, y \in X$ were an arbitrary pair satisfying $d(x,y) > n\cdot r$, we have that $\phi(a_1,\ldots,a_n) \in \mathbb{C}_u^{n\cdot r}[X]$ as was to be shown. $\square$

Remark 5.2.2. We now have most of the ingredients for the proof of the main theorem of this section. By Lemma 5.2.1 and Theorem 4.3.8 we know that

$$H_c^n(A:\ell) \cong H_c^n(A,\mathscr{B}:\ell) \cong H_c^n(A,\mathscr{B}).$$

In Sinclair and Smith [19] Theorem 3.3.1 they show that $H_c^n(A,\mathscr{B}) \cong H_c^n(\mathscr{B})$, which we also show in the sequel, Remark 6.3.6. Hence, by Theorem 1.0.2 $H_c^n(A:\ell) = 0$. Thus, we need only show that the homomorphism

$$H_c^n(A:\ell) \to H_c^n(A) \text{ induced by the inclusion } \mathscr{L}_c^n(A:\ell) \to \mathscr{L}_c^n(A)$$

is a surjection. By Lemma 4.3.2, averaging over the unitary group of $\ell^\infty(X)$, we know that for a cocycle $\phi \in \mathscr{L}_c^n(A)$, $(\phi - \partial K_n\phi) \in \mathscr{L}_c^n(A:\ell)$. Thus, to show that $H_c^n(A:\ell) \to H_c^n(A)$ is a surjection it suffices to show that

$K_n\phi \in \mathscr{L}_c^{n-1}(A)$ so that $\partial K_n\phi$ is a coboundary in $\mathscr{L}_c^n(A)$, for then

$$H_c^n(A : \ell) \ni [\phi - \partial K_n\phi] = [\phi] \text{ in } H_c^n(A).$$

Furthermore, since $H_w^n(A) \to H_c^n(A)$ is a surjection by the hypothesis of Theorem 5.0.1, we may assume that $\phi \in \mathscr{L}_w^n(A)$.

Before we set the notation to give a proof that if $H_w^n(A) \to H_c^n(A)$ is a surjection then $H_c^n(A) = 0$ for a general n we first show that if $H_w^2(A) \to H_c^2(A)$ is a surjection then $H_c^2(A) = 0$ since this low dimensional case is easier to follow but uses the same techniques as the higher dimensional cases.

Example 5.2.3. Let $\phi \in \mathscr{L}_w^2(A)$. Then $K_2\phi \in \mathscr{L}_c^1(A)$.

Proof. By the definition of K_n we have that

$$K_2 = J_1 - G_1 + G_1\partial J_1 \tag{32}$$

Since the other terms in this sum are handled similarly we only show that

$$G_1\partial J_1\phi \in \mathscr{L}_c^2(A).$$

Observe that for $a \in A$

$$G_1\partial J_1\phi(a) = \int_{\mathcal{U}} (\partial J_1\phi)(av, v^*)\,\mathrm{d}\,\mu(v)$$

$$= \int_{\mathcal{U}} av^*(J_1\phi)(v) + (J_1\phi)(av^*v) + (J_1\phi)(av^*)v \;\, \mathrm{d}\,\mu(v)$$

$$= \int_{\mathcal{U}} av^* \int_{\mathcal{U}} u^*\phi(u, v)\,\mathrm{d}\,\mu(u) + \int_{\mathcal{U}} u^*\phi(u, a)\,\mathrm{d}\,\mu(u) + \int_{\mathcal{U}} u^*\phi(u, av^*)v\,\mathrm{d}\,\mu(u)\,\mathrm{d}\,\mu(v)$$

$$= \int_{\mathcal{U}}\int_{\mathcal{U}} av^*u^*\phi(u, v)\,\mathrm{d}\,\mu(u)\,\mathrm{d}\,\mu(v) + \int_{\mathcal{U}}\int_{\mathcal{U}} u^*\phi(u, a)\,\mathrm{d}\,\mu(u)\,\mathrm{d}\,\mu(v) + \int_{\mathcal{U}}\int_{\mathcal{U}} u^*\phi(u, av^*)v\,\mathrm{d}\,\mu(u)\,\mathrm{d}\,\mu(v)$$

Where $\mathcal{U}$ is the unitary group of $\ell^\infty(X)$. Once more, since the other terms are handled similarly we only show that

$$\int_{\mathcal{U}}\int_{\mathcal{U}} u^*\phi(u, av^*)v \,\mathrm{d}\,\mu(u)\,\mathrm{d}\,\mu(v) \in \mathscr{L}_c^2(A).$$

Next, let p_x be the projection onto the span of the Dirac mass at x. Then, as in the derivations case, for any element f in the unit ball of $\ell^\infty(X)$ we may write f as the weakly convergent sum

$$f = \sum_{x\in X} \lambda_x p_x$$

Moreover, for any $\lambda \in \mathbb{D}^X$

$$\sum_{x\in X} \lambda_x p_x \text{ converges weakly to an element in the closed unit ball of } \ell^\infty(X).$$

Thus, for any fixed $\eta \in \mathbb{D}^X$ corresponding to an element g_η in the closed unit ball $(\ell^\infty(X))_1$ and $\lambda \in \mathbb{D}^X$, we have that

$$\sum_{x\in X} \lambda_x \phi(p_x, ag_\eta) \xrightarrow{\text{WOT}} \phi(f_\lambda, ag_\eta).$$

Likewise,

$$\sum_{x\in X} \lambda_x \phi(g_\eta, ap_x) \xrightarrow{\text{WOT}} \phi(g_\eta, af_\lambda),$$

and so $\phi(p_x, ap_y)$ is 2 separately symmetrically summable. Hence, by Corollary 5.1.2, given $\epsilon > 0$ there exists an $r \geq 0$ for all u, v in the unitary group of $\ell^\infty(X)$ such that

$$u^*\phi(u, av^*)v = a(u, v) + b(u, v)$$

where $a(u, v) \in \mathbb{C}_u^r[X]$ and $\|b(u, v)\| < \epsilon$. Recall that, by Lemma 3.2.8, if e_{yx} is the standard matrix unit, $\mathrm{Tr}(e_{yx}a(u, v)) = a(u, v)_{xy} = 0$ whenever $d(x, y) > r$ for all u, v in the unitary group of $\ell^\infty(X)$. Thus, the linear functional defining $\int_{\mathcal{U}}\int_{\mathcal{U}} a(u,v)\,\mathrm{d}\,\mu(u)\,\mathrm{d}\,\mu(v)$ sends e_{yx} to zero whenever $d(x,y) > r$ and so,

$$\int_{\mathcal{U}}\int_{\mathcal{U}} a(u,v)\,\mathrm{d}\,\mu(u)\,\mathrm{d}\,\mu(v) \in \mathbb{C}_u^r\,[X]\,.$$

Moreover, since our averaging operator is contractive, so

$$\left\| \int_V \int_U b(u,v)\,\mathrm{d}\,\mu(u)\,\mathrm{d}\,\mu(v) \right\| < \epsilon.$$

Thus,

$$\int_{\mathcal{U}}\int_{\mathcal{U}} u^*\phi(u,av^*)v\,\mathrm{d}\,\mu(u)\,\mathrm{d}\,\mu(v) \in \mathscr{L}_c^1(A). \qquad \square$$

Before we embark on the proof that $H_c^n(A) = 0$ if the map $H_w^n(A) \to H_c^n(A)$ is a surjection for a general n we show some properties of the map K_n arising from its construction and set some notation.

Lemma 5.2.4. *K_n is the sum of $\sum_{k=1}^n 2^{k-1}$ terms (before applying the boundary operator), where the first term is J_1, the next terms are the n-alternating sum of the maps G_k, and the remaining terms for $n \geq 2$ are of the form*

$$G_{j_i}\partial \dots G_{j_1}\partial J_1 \ \ \textit{or} \ \ G_{j_i}\partial \dots G_{j_2}\partial G_{j_1} \ \ \textit{for} \ \ j_i > j_{i-1} > \cdots > j_1. \tag{33}$$

Proof. Since K_n is defined by $K_n = J_n$ where $J_{k+1} = J_k + (-1)^k(G_k - G_k\partial J_k)$ we will induct on k. The case where $k = 2$ is handled by line (32).

Next, let $D_k = (G_k - G_k\partial J_k)$, then

$$J_{k+1} = J_k + (-1)^k D_k$$

$$= J_{k-1} + (-1)^{k-1}D_{k-1} + (-1)^k D_k$$

$$= J_1 + \sum_{j=1}^{k}(-1)^j D_j$$

$$= J_1 + \sum_{i=1}^{k}(-1)^i G_i + \sum_{j=1}^{k}(-1)^{j+1} G_j \partial J_j$$

Note that, since $j \leq k$ for all j in the last summation, by inductive hypothesis our terms are of the form of line (33).

Lastly, using the recursive definition of J_{k+1} and letting $|J_{k+1}|$ be the number of terms of J_{k+1},we have

$$|J_{k+1}| = |J_k| + |G_k| + |G_k \partial J_k| = 2\,|J_k| + 1 = 2\sum_{j=1}^{k} 2^{j-1} + 1 = \sum_{j=1}^{k+1} 2^{j-1}$$

as was to be shown. □

Lemma 5.2.5. *Let $\phi \in \mathscr{L}^n(A)$ and let $(a_1, \ldots, a_n),\ a_i \in A$ be given. Then*

$$(G_{j_i}\partial \ldots G_{j_1}\partial J_1\phi)(a_1, \ldots, a_{n-1}) \ \text{ and } \ (G_{j_i}\partial \ldots G_{j_2}\partial G_{j_1}\phi)(a_1, \ldots, a_{n-1})$$

are both finite sums of terms of the form

$$\int_{\mathcal{U}} \cdots \int_{\mathcal{U}} \prod_{k=1}^{N}(c_{1,k}v_{1,k})\phi\Big(\prod_{k=1}^{N} c_{2,k}v_{2,k}, \ldots, \prod_{k=1}^{N} c_{n,k}v_{n,k}\Big)\prod_{k=1}^{N} c_{n+1,k}v_{n+1,k}\ \mathrm{d}\,\mu(u_{j_i})\ldots \mathrm{d}\,\mu(u_{j_1})$$

where each $c_{\ell,k}$ is fixed as one of the a_j's or 1, and $v_{\ell,k} \in \mathcal{U}$ the unitary group of $\ell^\infty(X)$. Additionally $N < \infty$.

Proof. Consider

$$(G_{j_i}\partial \ldots G_{j_1}\partial J_1\phi)(a_1, \ldots, a_{n-1}).$$

Observe that, after applying G_{j_i}, in the l'th coordinate we will have: $a_l,\ a_l u_{j_i}^*$, or u_{j_i}. Note that we may write this coordinate as $c_l v_l$ where c_l is fixed as 1 or a_l and $v_l = 1,\ u_{j_i}$, or $u_{j_i}^*$. Also note that $v_l \in \mathcal{U}$. Thus we may write,

$$(G_{j_i}\partial \ldots G_{j_1}\partial J_1\phi)(a_1, \ldots, a_{n-1})$$

$$= \int_{\mathcal{U}} (\partial G_{j_{i-1}} \ldots \partial G_{j_1}\partial J_1\phi)(c_1v_1, \ldots, c_nv_n)\ \mathrm{d}\,\mu(u_{j_i}). \tag{34}$$

Next, since our averaging operator is finitely additive and the boundary operator introduces a finite number of terms, we may 'bring in' the averaging operator to each term. Additionally, since the boundary operator just moves one of the arguments to the coordinate to the left, in front of, or behind the map, we may write (after reindexing) a typical term obtained from applying the boundary map in line (34) as

$$\int_{\mathcal{U}} c_0 v_0 (G_{j_{i-1}}\partial \ldots G_{j_1}\partial J_1\phi)(c_1v_1c_2v_2, \ldots, c_{2n}v_{2n}c_{2n+1}v_{2n+1})c_{2n+2}v_{2n+2} \ \mathrm{d}\,\mu(u_{j_i})$$

where $c_k \in \{1, a_1, \ldots, a_{n-1}\}$ and $v_k \in \mathcal{U}$ (note that the c_k's will be fixed differently for each term). Applying this process again it is not hard to see that after applying $G_{j_i}\partial G_{i-1}\partial$ we will have a finite sum of terms of the form

$$\int_{\mathcal{U}}\int_{\mathcal{U}} \prod_{k=1}^{4}(c_{0,k}v_{0,k})(G_{j_{i-2}} \ldots \partial J_1\phi)\Big(\prod_{k=1}^{4} c_{1,k}v_{1,k}, \ldots, \prod_{k=1}^{4} c_{n,k}v_{n,k}\Big)\prod_{k=1}^{4} c_{n+1,k}v_{n+1,k} \ \mathrm{d}\,\mu(u_{j_i})\,\mathrm{d}\,\mu(u_{j_{i-1}}).$$

Note that the application of the J_1 map does not change our technique and eventually this process must end. Thus, the conclusion holds and we are done. $\square$

Definition 5.2.6. For each term obtained in the previous lemma the set of $\{c_{l,k}\}$ is fixed for that term. We shall call this a fixing of ϕ.

Lemma 5.2.7. *Let $(a_1, \ldots, a_n)$, $a_i \in A$ and $\phi \in \mathscr{L}^n_w(A)$ be given. Consider*

$$y_0\phi(y_1, \ldots, y_n)y_{n+1}, \quad y_i = \prod_{j=1}^{N_i} c_j f_{(i,j)}, \quad \text{where } N_i < \infty \tag{35}$$

where $f_{(i,j)}$ is any element in $(\ell^\infty(X))_1$, and each $c_j = a_k$ or 1 is fixed. Then for all $\epsilon > 0$ there exists an $r > 0$ (depending on the fixing of ϕ) such that $y_0\phi(y_1, \ldots, y_n)y_{n+1}$ can be ϵ-r-approximated.

Proof. Let $p_x \in \mathscr{B}(\ell^2(X))$ be the rank one projection onto the span of the Dirac mass at x. For any element f in the unit ball of $\ell^\infty(X)$, we may write

f as a strongly (and so weakly) convergent sum

$$f = \sum_{x \in X} f(x) p_x. \tag{36}$$

Then, for an arbitrary i, j where $1 \leq i \leq N$ and $0 \leq j \leq n+1$ and $f_{(\ell,k)} \in (\ell^\infty(X))_1$ fixed whenever $\ell, k \neq i, j$, we have that

$$\sum_{x_j \in X} \lambda_{x_j}^{(j)} y_0 \phi \left(y_1, \ldots, \left(\prod_{k=1}^{j-1} c_k f_{(i,k)} \right) c_j p_{x_j} \left(\prod_{k=j+1}^{N_i} c_k f_{(i,k)} \right), \ldots, y_n \right) y_{n+1}$$

weakly converges to

$$y_0 \phi(y_1, \ldots, \prod_{k=1}^{N_i} c_k f_{(i,k)}, \ldots, y_n) y_{n+1}$$

Moreover, (35) is bounded above by $\|\phi\| \prod_{k=1}^{n} \|a_k\|$ for all $f_{(\ell,k)} \in (\ell^\infty(X))_1$. Hence, since the weak and ultraweak topologies coincide on norm bounded sets and $\phi \in \mathscr{L}_w^n(A)$, we have that, for each fixing of ϕ,

$$\prod_{k=1}^{N_0} (c_{0,k} p_{x_{(0,k)}}) \phi \Big(\prod_{k=1}^{N_1} c_{1,k} p_{x_{(1,k)}}, \ldots, \prod_{k=1}^{N_n} c_{n,k} p_{x_{(n,k)}} \Big) \prod_{k=1}^{N_{n+1}} c_{n+1,k} p_{x_{(n+1,k)}}$$

is separately symmetrically summable. Thus, by Corollary 5.1.2, for all $\epsilon > 0$ there exists an $r > 0$ (depending on the fixing of ϕ) such that $y_0 \phi(y_1, \ldots, y_n) y_{n+1}$ can be ϵ-r-approximated. $\square$

Lemma 5.2.8. *$K_n \phi \in \mathscr{L}_c^{n-1}(C_u^*(X))$ whenever $\phi \in \mathscr{L}_w^n(C_u^*(X))$, where K_n is constructed by averaging over the unitary group of $\ell^\infty(X)$.*

Proof. Let $\epsilon > 0$ be given.

By Lemmas 5.1.6 and 5.2.7, $K_n\phi(a_1, \dots, a_{n-1})$ is the finite sum of finite sums of terms of the form

$$\int_{\mathcal{U}_{j_i}} \cdots \int_{\mathcal{U}_{j_1}} y_0\phi(y_1, \dots, y_n)y_{n+1} \,\mathrm{d}\,\mu(u_{j_1}) \dots \mathrm{d}\,\mu(u_{j_i})$$

where each term is a different fixing of ϕ (in the sense of Definition 5.2.6). Using Lemma 5.2.7 we may write each of these terms as

$$= \int_{\mathcal{U}_{j_i}} \cdots \int_{\mathcal{U}_{j_1}} a(\overline{u}) + b(\overline{u}) \,\mathrm{d}\,\mu(u_{j_1}) \dots \mathrm{d}\,\mu(u_{j_i}) \tag{37}$$

where each $a(\overline{u}) \in \mathbb{C}_u^r[X]$ and $\|b(\overline{u})\| < \epsilon/M$ for a given $M > 0$ and all $\overline{u} \in \mathcal{U}_{j_i} \times \cdots \times \mathcal{U}_{j_1}$. Thus, taking M and R sufficiently large, since $K_n\phi(a_1, \dots, a_{n-1})$ is the finite sum of terms as in line (37), $K_n\phi(a_1, \dots, a_{n-1})$ is ϵ-R-approximated. Since ϵ was arbitrary, we are done. $\square$

Proof of Theorem 5.0.1. By Remark 5.2.2, to show that $H_c^n(C_u^*(X)) = 0$ it suffices to show that $K_n\phi \in \mathscr{L}_c^{n-1}(C_u^*(X))$ whenever $\phi \in \mathscr{L}_w^n(C_u^*(X))$, which we have done in the previous lemma. $\square$

6 Ultraweak-Weak* Continuous Cohomology

In this section we discuss methods for relating norm continuous and ultraweak-weak* continuous cohomologies which will allow us to obtain the following result. If the norm continuous Hochschild cohomology of a uniform Roe algebra vanish in all dimensions then the ultraweak-weak* Hochschild cohomology of that uniform Roe algebra vanishes in all dimensions.

6.1 The Enveloping von Neumann Algebra

To accomplish the goals of this section we will have to use the "*enveloping von Neumann algebra*". To construct this algebra we begin by using the GNS construction to build a representation of our C*-algebra $\mathcal{A}$ on a Hilbert space $\{\pi_\tau, \mathcal{H}_\tau\}$ using a positive linear functional $\tau \in \mathcal{A}^*$ (cf. [13] Section 3.4). Next, let $S(\mathcal{A})$ be the state space of $\mathcal{A}$; that is, the space of all positive linear functionals of norm 1. We then form the *universal* representation of $\mathcal{A}$, $\{\pi, \mathcal{H}\}$ by

$$\pi := \bigoplus_{\tau \in S(\mathcal{A})} \pi_\tau \qquad \mathcal{H} := \bigoplus_{\tau \in S(\mathcal{A})} \mathcal{H}_\tau$$

The *enveloping von Neumann algebra* is then obtained by taking the ultraweak closure $\overline{\pi(\mathcal{A})}$ of $\pi(\mathcal{A})$ in $\mathscr{B}(\mathcal{H})$. Note that the weak closure of $\pi(\mathcal{A})$ coincides with the ultraweak closure of $\pi(\mathcal{A})$. Moreover, on norm bounded sets, the weak operator and ultraweak topologies coincide. Additionally, since $\overline{\pi(\mathcal{A})}$ is a von Neumann algebra it has a unique predual $\overline{\pi(\mathcal{A})}_*$ which we may identify with the ultraweakly continuous linear functionals on $\overline{\pi(\mathcal{A})}$.

Lemma 6.1.1 ([20] III.2.2, III.2.4)**.** *Let $\mathcal{A}$ be a C^*-algebra and $\{\pi, \mathcal{H}_\pi\}$ be the universal representation of $\mathcal{A}$. Then there is a unique linear map $\tilde{\pi}$ of the double dual $\mathcal{A}^{**}$ onto $\overline{\pi(\mathcal{A})}$ with the following properties:*

(i) If ι is the natural embedding from Definition 2.2.4 then the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\pi"] \arrow[d, "\iota"'] & \overline{\pi(\mathcal{A})} \\ \mathcal{A}^{**} \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

is commutative.

(ii) $\tilde{\pi}$ is $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-ultraweak continuous.

(iii) $\tilde{\pi}$ maps the unit ball $(\mathcal{A}^{**})_1$ onto the unit ball $(\overline{\pi(\mathcal{A})})_1$.

(iv) $\tilde{\pi}$ is a $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-ultraweak homeomorphism.

Proof. Define the maps

$$\pi_* \begin{cases} \overline{\pi(\mathcal{A})}_* \to \mathcal{A}^* \\ f \mapsto f \circ \pi \end{cases} \quad \text{and} \quad \tilde{\pi} \begin{cases} \mathcal{A}^{**} \to (\overline{\pi(\mathcal{A})}_*)^* \\ g \mapsto g \circ \pi_* \end{cases} .$$

Also, given $a \in \mathcal{A}$, let $\hat{a}$ be the image of a under the inclusion map ι. Next, given $a \in \mathcal{A}$ for all $f \in \overline{\pi(\mathcal{A})}_*$ we have

$$\langle f, (\tilde{\pi} \circ \iota)(a) \rangle = \langle f, \tilde{\pi}(\hat{a}) \rangle = \langle f, \hat{a} \circ \pi_* \rangle = \langle f \circ \pi, \hat{a} \rangle = \langle f, \pi(a) \rangle$$

Thus, $\tilde{\pi} \circ \iota = \pi$ on $\mathcal{A}$ and the diagram commutes.

For (ii), let $\{x_\alpha\} \subseteq \mathcal{A}^{**}$ be a net converging to $x \in \mathcal{A}^{**}$ in the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$ topology. Thus, given any $f \in \mathcal{A}^*$ and any $\epsilon > 0$ there exists an α_f such that

$$|\langle (x_\alpha - x), f\rangle| < \epsilon \ \text{ whenever } \ \alpha \geq \alpha_f$$

Next, given $g \in \overline{\pi(\mathcal{A})}_*$, we have that $\pi_*(g) \in \mathcal{A}^*$. Thus,

$$|\langle \tilde{\pi}(x_\alpha - x), g\rangle| = |\langle (x_\alpha - x), \pi_*(g)\rangle| < \epsilon \ \text{ whenever } \ \alpha \geq \alpha_{\pi_*(g)}$$

and so $\tilde{\pi}$ is $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-ultraweakly continuous.

For (iii), observe that the image $\tilde{\pi}((\mathcal{A}^{**})_1)$ of $(\mathcal{A}^{**})_1$ is ultraweakly compact by the continuity of $\tilde{\pi}$ and contains $\pi(\mathcal{A}_1)$. Moreover, since π is faithful, π is an isometry. Thus, $\pi(\mathcal{A}_1) = (\pi(\mathcal{A}))_1$, and so by the Kaplansky density theorem (cf. [13] Theorem 4.3.3)

$$\tilde{\pi}((\mathcal{A}^{**})_1) \supseteq \overline{\pi(\mathcal{A}_1)} = \overline{\pi(\mathcal{A})_1} = \left(\overline{\pi(\mathcal{A})}\right)_1.$$

For (iv) we first show that the map π_* defined above is a surjection. Let $\omega \in \mathcal{A}^*$. By the Jordan decomposition theorem (cf. [20] III.2.1), since π is the universal representation of $\mathcal{A}$, there exists $\xi_\omega, \eta_\omega \in \mathcal{H}_\pi$ such that

$$\omega(a) = \langle \pi(a)\xi_\omega, \eta_\omega\rangle \ \text{ for all } \ a \in \mathcal{A}.$$

On the other hand, by [14] Proposition 4.6.11, for any fixed $\xi, \eta \in \mathcal{H}_\pi$ the map

$$f : \overline{\pi(\mathcal{A})} \to \mathbb{C} \ \text{ defined by } \ x \mapsto \langle x\xi, \eta\rangle$$

is ultraweakly continuous and so $f \in \overline{\pi(\mathcal{A})}_*$. Hence, π_* is surjective.

Next, suppose that $\tilde{\pi}(g) = 0$. Then, $g \circ \pi_*$ is the zero map; that is, for all $f \in \overline{\pi(\mathcal{A})}_*$ we have

$$g \circ \pi_* \circ f = g \circ f \circ \pi = 0.$$

Thus, since π_* is surjective, $g(\mathcal{A}^*) = 0$ so that $\tilde{\pi}$ is injective.

Combining this result with part (iii) we see that $\tilde{\pi}$ is a bijection so it suffices to show that $\tilde{\pi}^{-1}$ is ultraweakly-$\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$ continuous. Let $\{x_\alpha\} \subseteq \overline{\pi(\mathcal{A})}$ be a net converging to $x \in \overline{\pi(\mathcal{A})}$. Recall that given any $\omega \in \mathcal{A}^*$ there exists an $f \in \overline{\pi(\mathcal{A})}_*$ such that $\omega = \pi_* \circ f$. Thus,

$$\left|\left\langle \tilde{\pi}^{-1}(x_\alpha - x), \omega \right\rangle\right| = \left|\left\langle \tilde{\pi}^{-1}(x_\alpha - x), \pi_*(f) \right\rangle\right| = |\langle (x_\alpha - x), f \rangle|$$

so $\tilde{\pi}$ is a $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-ultraweak homeomorphism. □

6.2 Weak Extension

In this subsection we discuss a method to extend separately continuous multilinear maps to separately ultraweakly continuous multilinear maps.

Lemma 6.2.1 ([19] 3.3.2)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras acting nondegenerately on a Hilbert space $\mathcal{H}$ with ultraweak closures $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$. Let τ be a bounded bilinear form on $\mathcal{A} \times \mathcal{B}$. If τ is separately ultraweakly continuous, then τ extends uniquely to to a separately ultraweakly continuous bilinear form $\overline{\tau}$ on $\overline{\mathcal{A}} \times \mathcal{B}$.*

Proof. For each fixed $b \in \mathcal{B}$ the ultraweakly continuous linear functional $\tau_b : a \mapsto \tau(a, b)$ extends uniquely to an ultraweakly continuous functional on $\overline{\mathcal{A}}$, say T_b. By the Kaplansky density theorem τ_b extends to T_b without change in norm. Then the mapping $T : \mathcal{B} \to (\overline{\mathcal{A}})_*$ defined by $b \mapsto T_b$ is a bounded linear map since

$$\|T\| = \sup_{\|b\|=1} \|T_b\| = \sup_{\|b\|=1} \sup_{\|a\|=1} \|T_b(a)\| = \|\tau\|.$$

Next, let $\{b_i\}_{i \in I}$ be a net in $\mathcal{B}$ converging ultraweakly to $b \in \mathcal{B}$. Then for any $a \in \mathcal{A}$, $\tau(a, b_i) \to \tau(a, b)$ in $\mathbb{C}$ since τ is separately ultraweakly continuous. Thus, T is ultraweak-$\sigma((\overline{\mathcal{A}})_*, \mathcal{A})$ continuous. Let $\mathcal{B}_1$ be the closed unit ball of $\mathcal{B}$. Then, by [1] Corollary II.9, since $T(\mathcal{B}_1)$ is bounded by $\|\tau\|$, $T(\mathcal{B}_1)$ is a relatively compact subset of $(\overline{\mathcal{A}})_*$ in the $\sigma((\overline{\mathcal{A}})_*, \overline{\mathcal{A}})$ topology. Thus, $T \restriction_{\mathcal{B}_1}$ is an ultraweak-$\sigma((\overline{\mathcal{A}})_*, \overline{\mathcal{A}})$ continuous map. Hence, for each fixed $a \in \overline{\mathcal{A}}$, the linear functional $b \mapsto \langle a, Tb \rangle$ is ultraweakly continuous on $\mathcal{B}_1$, and so on $\mathcal{B}$. Thus, we may define a form $\overline{\tau}$ with the desired properties on $\overline{\mathcal{A}} \times \mathcal{B}$ by

$$\overline{\tau}(a, b) = \langle a, Tb \rangle . \qquad \square$$

Lemma 6.2.2 ([19] Lemma 3.3.3)**.** *Let $\mathcal{A}$ be a C^*-algebra acting nondegenerately on a Hilbert space $\mathcal{H}$ and let $\mathcal{V}$ be the dual of a Banach space $\mathcal{V}_*$. If ϕ is a bounded n-linear map from $\mathcal{A}^n$ to $\mathcal{V}$ that is separately ultraweak-weak* continuous, then ϕ extends uniquely without change in norm to a bounded n-linear map $\overline{\phi}$ from $(\overline{\mathcal{A}})^n$ to $\mathcal{V}$ that is seperately ultraweak-weak* continuous.*

Proof. We first prove this for $\mathcal{V} = \mathbb{C}$. We begin by constructing a finite sequence of n-linear functionals where $\phi_0 = \phi$ and $\phi_k : (\overline{\mathcal{A}})^k \times \mathcal{A}^{n-k}$ uniquely extends ϕ_{k-1} without change of norm to a map that is separately ultraweakly continuous. Note that uniqueness follows from ultraweak continuity and that $\mathcal{A}$ is ultraweakly dense in $\overline{\mathcal{A}}$. For the base case, letting a_1 vary and holding the remaining a_k's steady we obtain an ultraweakly continuous linear functional on $\mathcal{A}$ defined by

$$a_1 \mapsto \phi(a_1, \ldots, a_n).$$

Thus, by the Kaplansky density theorem, this functional can be uniquely extended without change in norm to an ultraweakly continuous functional on $\overline{\mathcal{A}}$ denoted by

$$a_1 \mapsto \phi_1(a_1, \ldots, a_n).$$

Then, ϕ_1 is a separately ultraweakly continuous n-linear form from $\overline{\mathcal{A}} \times \mathcal{A}^{n-1}$ to $\mathbb{C}$ that extends ϕ without change in norm. This concludes the base case.

The first part of the inductive step proceeds the same as the base case by interchanging the roles of a_1 with a_k assuming that the maps $\phi_1, \ldots, \phi_{k-1}$ have been constructed so that $\phi_j : (\overline{\mathcal{A}})^j \times \mathcal{A}^{n-j}$ uniquely extends ϕ_{j-1} ($1 \leq j \leq k-1$) without change of norm to a map that is separately ultraweakly continuous. What remains to be shown is the ultraweak continuity of ϕ_k in its other arguments when $a_k \in \overline{\mathcal{A}} \setminus \mathcal{A}$ is fixed. To that end, let $1 \leq j \leq n$, and fix a_i whenever $i \neq j, k$. Let $\mathcal{B} = \overline{\mathcal{A}}$ if $j < k$ and let $\mathcal{B} = \mathcal{A}$ if $j > k$. Then we may define τ to be the bounded bilinear form on $\mathcal{A} \times \mathcal{B}$ by

$$\tau(a_k, a_j) = \phi_{k-1}(a_1, \ldots, a_n).$$

This form is separately ultraweakly continuous by inductive hypothesis and so extends to a bounded bilinear form $\overline{\tau}$ on $\overline{\mathcal{A}} \times \mathcal{B}$ which is separately ultraweakly continuous by Lemma 6.2.1. Moreover, by the ultraweak continuity of $\overline{\tau}$ and ϕ_k for $a_k \in \overline{\mathcal{A}}$ and since they agree on $\mathcal{A}$ we have

$$\overline{\tau}(a_k, a_j) = \phi_k(a_1, \ldots, a_n) \text{ on } \overline{\mathcal{A}} \times \mathcal{B}.$$

Thus, ϕ_k is separately ultraweakly continuous in the jth position when $a_k \in \overline{\mathcal{A}}$ is fixed.

Next, let $\mathcal{V}$ be the dual of a Banach space $\mathcal{V}_*$ and $\phi : \mathcal{A}^n \to \mathcal{V}$. Then, for each $w \in \mathcal{V}_*$ we may define a separately ultraweakly continuous n-linear form on $\mathcal{A}^n$ by

$$\tau_w(a_1, \ldots, a_n) = \langle \phi(a_1, \ldots, a_n), w \rangle, \quad \|\tau_w\| \leq \|\phi\| \, \|w\|.$$

Then, from what we have shown above, τ_w extends without change of norm to a separately ultraweakly continuous n-linear form $\overline{\tau_w}$ on $(\overline{\mathcal{A}})^n$. Thus, we may define a bounded linear functional on $\mathcal{V}_*$ by

$$\overline{\phi}(a_1, \ldots, a_n) : w \mapsto \overline{\tau_w}(a_1, \ldots, a_n)$$

By construction $\overline{\phi}$ is a separately ultraweak-weak* n-linear map from $(\overline{\mathcal{A}})^n$ to $\mathcal{V} = (\mathcal{V}_*)^*$ where $\|\overline{\phi}\| = \|\phi\|$. □

6.3 A Bridge Between Ultraweak-Weak* Continuous and Norm Continuous Cohomology

The following lemma can be found in Blackadar [4] III.5.2.11 or Takesaki [20] III.2.4, III.2.14.

Lemma 6.3.1. *Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space $\mathcal{H}$ with weak closure $\overline{\mathcal{A}}$. If π is the universal representation of $\mathcal{A}$, then there is a projection p in the center of the weak closure $\overline{\pi(\mathcal{A})}$ of $\pi(\mathcal{A})$ and a $*$-isomorphism*

$$\theta : p\overline{\pi(\mathcal{A})} \to \overline{\mathcal{A}} \ \ \textit{such that}$$

$$\theta(p\pi(a)) = a \ \ \textit{and} \ \ \theta(px) = \pi^{-1}(x) \ \textit{for all} \ a \in \mathcal{A} \ \textit{and} \ x \in \pi(\mathcal{A}). \tag{38}$$

Moreover, θ is a homeomorphism from $p\overline{\pi(\mathcal{A})}$ onto $\overline{\mathcal{A}}$ if both have their ultraweak topologies, since $*$-isomorphisms between von Neumann algebras are ultraweak homeomorphisms.

Lemma 6.3.2 ([19] Lemma 3.3.4)**.** *Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space $\mathcal{H}$ with weak closure $\overline{\mathcal{A}}$. Let π be the universal representation of $\mathcal{A}$, and let p, θ be as in Lemma 6.3.1. Additionally, let $\mathcal{V}$ be a dual normal $\overline{\mathcal{A}}$-module. Then $\mathcal{V}$ may be regarded as a dual normal $\overline{\pi(\mathcal{A})}$-module via*

$$x \cdot v = \theta(px)v \ \ \textit{and} \ \ v \cdot x = v\theta(px) \tag{39}$$

and there are continuous linear maps

$$T_n : \mathscr{L}_c^n(\mathcal{A}, \mathcal{V}) \to \mathscr{L}_w^n(\overline{\pi(\mathcal{A})}, \mathcal{V}),$$

$$S_n : \mathscr{L}_w^n(\overline{\pi(\mathcal{A})}, \mathcal{V}) \to \mathscr{L}_w^n(\overline{\mathcal{A}}, \mathcal{V})$$

$$W_n : \mathscr{L}_w^n(\overline{\pi(\mathcal{A})}, \mathcal{V}) \to \mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$$

such that:

(i) $\partial T_n = T_{n+1}\partial$, $\partial W_n = W_{n+1}\partial$, *and* $\partial S_n = S_{n+1}\partial$,

(ii) $\|T_n\|, \|S_n\|, \|W_n\| \leq 1$,

(iii) if $\mathcal{B}$ *is a* C^**-subalgebra of* $\mathcal{A}$, T_n *maps* $\mathcal{B}$*-multimodular maps to* $\overline{\pi(\mathcal{B})}$*-multimodular maps, and* S_n *and* W_n *map* $\overline{\pi(\mathcal{B})}$*-multimodular maps to* $\mathcal{B}$*-multimodular maps,*

(iv) S_nT_n *is a projection from* $\mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$ *onto* $\mathscr{L}_w^n(\mathcal{A}, \mathcal{V})$,

(v) if $\mathcal{C}$ *is the* C^**-subalgebra of* $\overline{\pi(\mathcal{A})}$ *generated by* 1 *and* p, *and if*

$$\psi \in \mathscr{L}_w^n(\overline{\pi(\mathcal{A})}, \mathcal{V} : \mathcal{C}),$$

then $W_n\psi = S_n\psi \in \mathscr{L}_w^n(\overline{\mathcal{A}}, \mathcal{V})$,

(vi) W_nT_n *is the identity map on* $\mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$.

Note that for $\psi \in \mathscr{L}_w^n(\overline{\pi(\mathcal{A})}, \mathcal{V} : \mathcal{C})$ being $\mathcal{C}$-multimodular is equivalent to having the property that $\psi(a_1, \ldots, a_n) = 0$ if any of the arguments $a_j \in (1-p)\overline{\pi(\mathcal{A})}$.

Proof of the properties of the map T_n. We begin by constructing T_n. Let $\phi \in \mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$. Then the equation

$$\phi_1(x_1, \ldots, x_n) = \phi(\theta(px_1), \ldots, \theta(px_n)) \text{ for all } x_1, \ldots, x_n \in \pi(\mathcal{A}) \tag{40}$$

defines $\phi_1 \in \mathscr{L}_c^n(\pi(\mathcal{A}), \mathcal{V})$. Moreover, $\phi \mapsto \phi_1$ is an isometry by line (38).

Next, let $v \in \mathcal{V}_*$ and define a map

$$\tau_{v,i} : \pi(\mathcal{A}) \to \mathbb{C} \text{ by } x_i \mapsto \langle v, \phi_1(x_1, \ldots, x_i, \ldots, x_n)\rangle$$

where the x_k's $k \neq i$ are fixed.

Clearly this map is bounded and so $\tau_{v,i} \in \pi(\mathcal{A})^*$. Then, since π is the universal representation of $\mathcal{A}$, by [20] III.2.4 every continuous linear functional in $\pi(\mathcal{A})^*$ is ultraweakly continuous and so $\tau_{v,i}$ is ultraweakly contiuous. Thus, ϕ_1 is separately ultraweak-weak* continuous since $v \in \mathcal{V}_*$ was arbitrary. Hence, $\phi_1 \in \mathscr{L}_w^n(\pi(\mathcal{A}), \mathcal{V})$. Moreover, by Lemma 6.2.2, we may uniquely extend ϕ_1 to $\overline{\phi_1} \in \mathscr{L}_w^n(\overline{\pi(\mathcal{A})}, \mathcal{V})$ without change of norm. The map T_n is then defined by $T_n\phi = \overline{\phi_1}$.

Next, for all $x_j,\ x_{j+1}$ in $\pi(\mathcal{A})$ we have that

$$\theta(px_j)\theta(px_{j+1}) = \theta(px_jpx_{j+1}) = \theta(px_jx_{j+1})$$

since θ is a $*$-isomorphism and p is a central projection in $\overline{\pi(\mathcal{A})}$. Thus, for all $x_1, \ldots, x_{n+1} \in \pi(\mathcal{A})$ we have

$$\partial T_n\phi(x_1, \ldots, x_{n+1})$$

$$= x_1(T_n\phi)(x_2, \ldots, x_{n+1})$$
$$+ \sum_{j=1}^{n} (-1)^j (T_n\phi)(x_1, \ldots, x_jx_{j+1}, \ldots, x_{n+1})$$
$$+ (-1)^{n+1}(T_n\phi)(x_1, \ldots, x_n)x_{n+1}$$

$$= \theta(px_1)\phi\big(\theta(px_2), \ldots, \theta(px_{n+1})\big)$$
$$+ \sum_{j=1}^{n} (-1)^j \phi\big(\theta(px_1), \ldots, \theta(px_j)\theta(px_{j+1}), \ldots, \theta(px_{n+1})\big)$$
$$+ (-1)^{n+1}\phi\big(\theta(px_1), \ldots, \theta(px_n)\big)\theta(px_{n+1})$$

$$= T_{n+1}\partial\phi(x_1, \ldots, x_{n+1}).$$

Then, since the maps $\partial T_n\phi$ and $T_{n+1}\partial\phi$ are separately ultraweakly-weak* continuous we have that

$$\partial T_n\phi = T_{n+1}\partial\phi.$$

Lastly, we show that T_n maps $\mathcal{B}$-multimodular maps to $\overline{\pi(\mathcal{B})}$-multimodular maps. Let $\phi \in \mathscr{L}_c^n(\mathcal{A}, \mathcal{V} : \mathcal{B})$, $x_1, \ldots, x_n \in \pi(\mathcal{A})$, and $b \in \mathcal{B}$. Then, since $x_j\pi(b) \in \pi(\mathcal{A})$ we have

$$T_n\phi(x_1, \ldots, x_j\pi(b), \ldots, x_n) = \phi_1(x_1, \ldots, x_j\pi(b), \ldots, x_n).$$

Next, setting $x_j = \pi(a_j)$,

$$\phi_1(x_1, \ldots, x_j\pi(b), \ldots, x_n) = \phi\big(\theta(px_1), \ldots, \theta(px_j)\theta(p\pi(b)), \ldots, x_n\big)$$

$$= \phi(a_1, \ldots, a_jb, a_{j+1}, \ldots, a_n) = \phi(a_1, \ldots, a_j, ba_{j+1}, \ldots, a_n)$$

$$= \phi\big(\theta(px_1), \ldots, \theta(px_j), \theta(p\pi(b))\theta(px_{j+1}) \ldots, x_n\big)$$

$$= T_n\phi(x_1, \ldots, x_j, \pi(b)x_{j+1}, \ldots, x_n).$$

Then, using the action defined on line (39), a similar calculation shows that

$$\pi(b)T_n\phi(x_1, \ldots, x_j, \ldots, x_n) = T_n\phi(\pi(b)x_1, \ldots, x_j, \ldots, x_n)$$

and that

$$T_n\phi(x_1, \ldots, x_j, \ldots, x_n\pi(b)) = T_n\phi(x_1, \ldots, x_j, \ldots, x_n)\pi(b).$$

Hence, noting that T_n is ultraweak-weak* continuous and that $\mathcal{V}$ is a normal $\overline{\mathcal{A}}$-bimodule, T_n maps $\mathcal{B}$-multimodular maps to $\overline{\pi(\mathcal{B})}$-multimodular maps. $\square$

Proof of the properties of the map S_n. We begin by constructing S_n. The map $S_n : \mathscr{L}_w^n(\overline{\pi(\mathcal{A})}, \mathcal{V}) \to \mathscr{L}_w^n(\overline{\mathcal{A}}, \mathcal{V})$ is defined by

$$(S_n\psi)(a_1, \dots, a_n) = \psi(\theta^{-1}(a_1), \dots, \theta^{-1}(a_n)).$$

Note that, since θ is a isomomorphism, $\|S_n\| \leq 1$.

Observe that, by (38) and the action defined on line (39)

$$\begin{aligned} S_{n+1}\partial\psi(a_1, \dots, a_{n+1}) &= \partial\psi(\theta^{-1}(a_1), \dots, \theta^{-1}(a_{n+1})) \\ &= a_1\psi(\theta^{-1}(a_2), \dots, \theta^{-1}(a_{n+1})) \\ &+ \sum_{j=1}^{n}(-1)^j\psi(\theta^{-1}(a_1), \dots, \theta^{-1}(a_ja_{j+1}), \dots, \theta^{-1}(a_{n+1})) \\ &+(-1)^{n+1}\psi(\theta^{-1}(a_1), \dots, \theta^{-1}(a_n))a_{n+1} \\ &= \partial S_n\psi(a_1, \dots, a_{n+1}) \end{aligned}$$

for all $(a_1, \dots, a_{n+1}) \in \mathcal{A}^{n+1}$. Now, since θ is an $*$-isomorphism between von Neumann algebras and every $\psi \in \mathscr{L}_w^n(\mathcal{A}, \mathcal{V})$ is ultraweak-weak* continuous, it follows that $S_n\psi$ is ultraweak-weak* continuous so that the previous calculation holds on $\overline{\mathcal{A}}$ and so $S_{n+1}\partial\psi = \partial S_n\psi$.

Next, using that θ is a $*$-isomorphism once more, if $\mathcal{B}$ is a C*-subalgebra of $\mathcal{A}$, then $\theta^{-1}(\overline{\mathcal{B}}) = p\overline{\pi(\mathcal{B})}$. Thus, if ψ is a $\overline{\pi(B)}$-multimodular map then $S_n\psi$ is a $\overline{\mathcal{B}}$-multimodular map.

Lastly we show that S_nT_n is a projection from $\mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$ onto $\mathscr{L}_w^n(\mathcal{A}, \mathcal{V})$. Suppose that $\phi \in \mathscr{L}_w^n(\mathcal{A}, \mathcal{V})$. Note that S_n sends $T_n\phi$ to $\mathscr{L}_w^n(\mathcal{A}, \mathcal{V})$. However, ϕ is already in $\mathscr{L}_w^n(\mathcal{A}, \mathcal{V})$ and the extension that takes place in the construction of T_n is unique. Hence, for $a_i \in \mathcal{A}$ we have

$$S_nT_n\phi(a_1, \dots, a_n) = T_n\phi(\theta^{-1}(a_1), \dots, \theta^{-1}(a_n)) = \phi(a_1, \dots, a_n). \quad \square$$

Proof of the properties of the map W_n. The map $W_n : \mathscr{L}_w^n(\overline{\pi(\mathcal{A})}, \mathcal{V}) \to \mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$ is defined by

$$W_n\psi(a_1, \ldots, a_n) = \psi(\pi(a_1), \ldots, \pi(a_n)).$$

Note that, since π is a homomorphism, $\|W_n\| \leq 1$.

Observe that, $\partial W_n\psi(a_1, \ldots, a_{n+1})$

$$= a_1(W_n\psi)(a_2, \ldots, a_{n+1})$$

$$+\sum_{j=1}^{n}(-1)^j(W_n\psi)(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1})$$

$$+(-1)^{n+1}(W_n\psi)(a_1, \ldots, a_n)a_{n+1}$$

$$= \pi(a_1) \cdot \psi(\pi(a_2), \ldots, \pi(a_{n+1}))$$

$$+\sum_{j=1}^{n}(-1)^j\psi(\pi(a_1), \ldots, \pi(a_j)\pi(a_{j+1}), \ldots, \pi(a_{n+1}))$$

$$+(-1)^{n+1}\psi(\pi(a_1), \ldots, \pi(a_n)) \cdot \pi(a_{n+1})$$

$$= W_{n+1}\partial\psi(a_1, \ldots, a_{n+1})$$

by the definition of the action in line (39).

Next, if $\phi \in \mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$, then

$$W_nT_n\phi(a_1, \ldots, a_n) = T_n\phi(\pi(a_1), \ldots, \pi(a_n))$$

$$\phi(\theta(p\pi(a_1)), \ldots, \theta(p\pi(a_n))) = \phi(a_1, \ldots, a_n)$$

so that W_nT_n is the identity on $\mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$.

Lastly, if $\psi \in \mathscr{L}_w^n(\overline{\pi(\mathcal{A})}, \mathcal{V} : \mathcal{C})$ we have,

$$\begin{aligned}
W_n\psi(a_1,\ldots,a_n) &= \psi(\pi(a_1),\ldots,\pi(a_n)) && \\
&= \psi(\pi(a_1),\ldots,\pi(a_n))\cdot p && \text{by (39)} \\
&= \psi(p\pi(a_1),\ldots,p\pi(a_n)) && \text{since } p \text{ is a central projection} \\
& && \text{and } \psi \text{ is } \mathcal{C}\text{-multimodular} \\
&= \psi(\theta^{-1}(a_1),\ldots,\theta^{-1}(a_n)) && \text{since } \theta^{-1}(a) = p\pi(a) \\
&= S_n\psi(a_1,\ldots,a_n)
\end{aligned}$$

□

Remark 6.3.3. Note that, if $\mathcal{A} \subseteq \mathcal{V} = \mathscr{B}(\mathcal{H})$ then W_n maps $\mathscr{L}_w^n(\pi(\mathcal{A}), \mathcal{A})$ to $\mathscr{L}_c^n(\mathcal{A})$.

Lemma 6.3.4 ([19] Lemma 3.3.5)**.** *Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{V}$ be a dual normal $\mathcal{A}$-bimodule. Then the homomorphism*

$$H_w^n(\mathcal{A},\mathcal{V}) \to H_c^n(\mathcal{A},\mathcal{V}) \ \ \textit{induced by} \ \ \mathscr{L}_w^n(\mathcal{A},\mathcal{V}) \to \mathscr{L}_c^n(\mathcal{A},\mathcal{V})$$

is surjective.

Proof. Note that span $\{1, 2p-1\} = \mathcal{C}$. Moreover, the unitary group of $\mathcal{C}$ is $\mathcal{U} = \{\lambda p + \mu(1-p) : \ \lambda,\mu \in \mathbb{C}, |\lambda| = |\mu| = 1\}$. We will average over this group to build the map K_n from Lemma 4.3.2. Recall that whenever $u \in \mathcal{U}$, $u = \lambda p + \mu(1-p)$ appears during the construction of K_n so does $u^* = \overline{\lambda}p + \overline{\mu}(1-p)$. Thus, using the multilinearity of our maps, our averaging operator becomes a two term sum and so preserves weak continuity. Thus, for each

$$\psi \in \mathscr{L}_w^n(\overline{\pi(\mathcal{A})}, \mathcal{V}), \ \ K_n\psi \in \mathscr{L}_w^{n-1}(\overline{\pi(\mathcal{A})}, \mathcal{V}). \tag{41}$$

So if ψ is a cocycle, then $\psi - \partial K_n\psi$ is $\mathcal{C}$-multimodular.

Next, let $\phi \in \mathscr{L}_c^n(\mathcal{A},\mathcal{V})$ be a cocycle. Then, by Lemma 6.3.2 (i), we have that

$$\partial T_n\phi = T_{n+1}\partial\phi = 0$$

so that $T_n\phi$ is a cocycle in $\mathscr{L}_w^n(\overline{\pi(\mathcal{A})}, \mathcal{V})$. Hence, $(T_n\phi - \partial K_n T_n\phi)$ is $\mathcal{C}$-multimodular. Recall that, by Lemma 6.3.2 (v), W_n takes $\mathcal{C}$-multimodular maps to ultraweak-weak* continuous maps and so

$$W_n(T_n\phi - \partial K_n T_n\phi) \in \mathscr{L}_w^n(\overline{\mathcal{A}}, \mathcal{V}).$$

Next, for $\phi \in \mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$, by line (41) we have that $K_n T_n\phi \in \mathscr{L}_w^{n-1}(\overline{\pi(\mathcal{A})}, \mathcal{V})$. Thus, by the definition of W_n, it follows that

$$\partial W_{n-1} K_n T_n \phi \text{ is a coboundary in } \mathscr{L}_c^n(\mathcal{A}, \mathcal{V}).$$

Additionally, by Lemma 6.3.2 (vi) $W_n T_n$ is the identity map on $\mathscr{L}_c^n(\mathcal{A}, \mathcal{V})$ and so

$$W_n(T_n\phi - \partial K_n T_n\phi) = (\phi - \partial W_{n-1} K_n T_n \phi) \in \mathscr{L}_w^n(\overline{\mathcal{A}}, \mathcal{V}).$$

Thus, the map $H_w^n(\mathcal{A}, \mathcal{V}) \to H_c^n(\mathcal{A}, \mathcal{V})$ is a surjection. $\square$

Theorem 6.3.5 ([19] Theorem 3.3.1)**.** *Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space $\mathcal{H}$ with weak closure $\overline{\mathcal{A}}$. Additionally, let $\mathcal{V}$ be a dual normal $\overline{\mathcal{A}}$-module. Then,*

$$H_c^n(\mathcal{A}, \mathcal{V}) \cong H_w^n(\mathcal{A}, \mathcal{V}) \cong H_w^n(\overline{\mathcal{A}}, \mathcal{V})$$

Proof. By the previous lemma we have that the map $H_w^n(\mathcal{A}, \mathcal{V}) \to H_c^n(\mathcal{A}, \mathcal{V})$ is a surjection. To see that this map is injective first note that for $\psi \in \mathscr{L}_c^{n-1}(\mathcal{A}, \mathcal{V})$ we have that $S_{n-1}T_{n-1}\psi \in \mathscr{L}_w^{n-1}(\mathcal{A}, \mathcal{V})$ by Lemma 6.3.2 (iv). Next, if $\phi \in \mathscr{L}_w^n(\mathcal{A}, \mathcal{V})$ with $\phi = \partial\psi$ where $\psi \in \mathscr{L}_c^{n-1}(\mathcal{A}, \mathcal{V})$, then

$$\phi = \partial\psi = S_n T_n \partial\psi = \partial S_{n-1} T_{n-1} \psi.$$

Thus, the map $H_w^n(\mathcal{A}, \mathcal{V}) \to H_c^n(\mathcal{A}, \mathcal{V})$ induced by the inclusion is also injective and so an isomorphism.

Lastly, by Lemma 6.2.2, the restriction map $\mathscr{L}_w^n(\overline{\mathcal{A}}, \mathcal{V}) \to \mathscr{L}_w^n(\mathcal{A}, \mathcal{V})$ is an isomorphism and so we are done. $\square$

Remark 6.3.6. Note that the ultraweak closure of $C_u^*(X)$ is $\mathscr{B}(\ell^2(X))$. Hence, by the previous theorem we have that

$$H_c^n(C_u^*(X), \mathscr{B}(\ell^2(X))) \cong H_c^n(\mathscr{B}(\ell^2(X))).$$

6.4 On the Vanishing of the Ultraweak-Weak* Continuous Cohomology of Uniform Roe Algebras

In this subsection we prove the following theorem.

Theorem 6.4.1. *If the continuous Hochschild cohomology of a uniform Roe algebra associated to a bounded geometry metric space vanish in all dimensions, then the ultraweak-weak* continuous Hochschild cohomology of that uniform Roe algebra vanishes in all dimensions also.*

Before we prove this theorem we will need a few lemmas. Once more for notational convenience throughout we let: $A = C_u^*(X)$, $\mathscr{B} = \mathscr{B}(\ell^2(X))$, and $\ell = \ell^\infty(X)$.

Lemma 6.4.2. *Let π be the universal representation of A, and let p be the projection from Lemma 6.3.1. If $\{q_\alpha\}$ is the net of finite rank projections in ℓ with its usual ordering then*

$$\pi(q_\alpha) \xrightarrow{\text{ultraweakly}} p \ \ in \ \ \overline{\pi(\ell)}.$$

Proof. Recall that the double dual of the compact operators $\mathfrak{K}(\mathcal{H})^{**}$ is naturally identified with $\mathscr{B}$ (cf. [20] II.1.8). Moreover, since $\mathfrak{K}(\mathcal{H})$ is an ideal in $C_u^*(X)$ and $\{q_\alpha\}$ is an approximate unit for $\mathfrak{K}(\mathcal{H})$, by Blackadar [4] III.5.2.11, there exists a central projection $q \in A^{**}$ such that

$$\hat{q}_\alpha \to q \ \text{ in the } \ \sigma(A^{**}, A^*) \ \text{ topology and } \ qA^{**} = \mathfrak{K}(\mathcal{H})^{**} \cong p\overline{\pi(A)}$$

Thus, if $\tilde{\pi}$ is the map from Lemma 6.1.1, using Lemma 6.1.1, we have that $\tilde{\pi}(q) = p$. Moreover, since $\{q_\alpha\} \subseteq \ell$ and $\tilde{\pi}$ is a $\sigma(A^{**}, A^*)$-ultraweak homeomorphism, we have that

$$\pi(q_\alpha) \xrightarrow{\text{ultraweakly}} p \quad \text{and} \quad p \in \overline{\pi(\ell)}. \quad \square$$

Lemma 6.4.3. *If $\phi \in \mathscr{L}_c^n(A : \ell)$ then $\phi \in \mathscr{L}_w^n(A)$. That is, $\mathscr{L}_c^n(A : \ell) = \mathscr{L}_w^n(A : \ell) \subseteq \mathscr{L}_w^n(A)$.*

Proof. Since T_n takes ℓ-multimodular maps to $\overline{\pi(\ell)}$-multimodular maps we have

$$\begin{aligned}
S_nT_n\phi(a_1, \ldots, a_n) &= T_n\phi(\theta^{-1}(a_1), \ldots, \theta^{-1}(a_n)) && \text{by the definition of } S_n \\
&= T_n\phi(p\pi(a_1), \ldots, p\pi(a_n)) && \text{by the properties of } \theta \\
&= p \cdot T_n\phi(\pi(a_1), \ldots, \pi(a_n)) && \text{since } p \text{ is central and} \\
& && T_n\phi \text{ is } \overline{\pi(\ell)}\text{-multimodular.} \\
&= p \cdot \phi(a_1, \ldots, a_n) && \text{by the definition of } T_n. \\
&= \phi(a_1, \ldots, a_n) && \text{by line (39).}
\end{aligned}$$

Then, since S_nT_n is a projection from $\mathscr{L}_c^n(A)$ onto $\mathscr{L}_w^n(A, \mathscr{B})$, we are done. $\square$

Lemma 6.4.4. *If $H_c^{n-1}(A) = 0$ then the map $H_w^n(A) \to H_c^n(A)$ is an injection.*

Proof. Suppose that $\phi \in \mathscr{L}_w^n(A)$ with $\phi = \partial\psi$ for some $\psi \in \mathscr{L}_c^{n-1}(\mathcal{A})$. So if $H_c^{n-1}(A) = 0$, we have $H_c^{n-1}(A : \ell) \cong H_c^{n-1}(A)$, since by Remark 5.2.2 $H_c^{n-1}(A : \ell) = 0$. Hence, without loss of generality, we may assume $\psi \in \mathscr{L}_c^{n-1}(A : \ell) \subseteq \mathscr{L}_w^{n-1}(\mathcal{A})$. Thus, $[\phi] = 0$ in $H_w^n(A)$ and we are done. $\square$

We are now ready to prove Theorem 6.4.1.

Proof of Theorem 6.4.1. Since derivations are automatically weakly continuous by Theorem 3.1.3, $H^1_w(A) = H^1_c(A) = 0$. Next, given any $n > 1$ we have that $H^{n-1}_c(A) = 0$, so by Lemma 6.4.4, $H^n_w(A) \to H^n_c(A)$ is an injection. Moreover, $H^n_c(A) = 0$, so we must have that $H^n_w(A) = 0$. Since n was arbitrary we are done. $\square$

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